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In: Zdeněk Frolík (ed.): Abstracta. 7th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1979. pp. 92--94.

Persistent URL: http://dml.cz/dmlcz/701156

SEVENTH WINTER SCHOOL (1979)

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A *topological transformation group (ttg)* is a triple $\langle G,X,\pi\rangle$ where G is a topological group, X is a topological space and π is an *action* of G on X, that is, $\pi: G \times X \to X$ is a continuous mapping such that
(i) $\forall x \in X: \pi(e,x) = x$ (e denotes the identity element of G),
(ii) $\forall (s,t,x) \in G \times G \times X: \pi(t,\pi(s,x)) = \pi(ts,x)$.
A ttg $\langle G,X,\pi\rangle$ will also be called a G-space. Using the notation $\pi^t x := \pi(t,x)$ for $(t,x) \in G \times X$, we have $\pi^e = 1_X$ and $\pi^t \circ \pi^s = \pi^{ts}$. So $t \mapsto \pi^t$ defines a homomorphism of G into the group of all autohomeomorphisms of X.

If $\langle G,X,\pi\rangle$ is a G-space and if there exists a G-space $\langle G,Y,\sigma\rangle$ such that Y is a compact Hausdorff space and X is (homeomorphic with) a dense subset of Y such that $\sigma^t|_X = \pi^t$ for every $t \in G$, then $\langle G,Y,\sigma\rangle$ is called a *G-compactification of* $\langle G,X,\pi\rangle$. A necessary condition for $\langle G,X,\pi\rangle$ to have a G-compactification is, that X is a Tychonoff space. The question is, whether of this condition is sufficient. For $\mathbb{R}$-spaces, this question occurs in [2], and for general G-spaces in [3] in the context of reflection of G-spaces in the category of compact Hausdorff G-spaces. The following result, which has just been published [4], provides a partial solution to this problem:

THEOREM 1. *If G is locally compact then every G-space $\langle G,X,\pi\rangle$ with X a Tychonoff space has a G-compactification.*

Another partial solution is included in Theorem 3 below, which is joint work of H. LUDESCHER (Timisoara, Romania) and myself. It is a consequence of the following theorem, in which the following notation will be used: $\pi_x t := \pi(t,x) = \pi^t x$ for $(t,x) \in G \times X$; so $\pi_x: G \to X$ is continuous for all $x \in X$.

THEOREM 2. (BROOK, [1]). *Let $\langle G,X,\pi\rangle$ be a G-space in which X is a Tychonoff space, and suppose that there exists a uniformity $\mathcal{U}$ for X (i.e. compatible with the topology of X) such that the following conditions are satisfied:*
(i) $\{\pi_x : x \in X\}$ *is $\mathcal{U}$-equicontinuous at e;*
(ii) $\forall t \in G: \pi^t: X \to X$ *is $\mathcal{U}$-uniformly continuous.*
Then $\langle G,X,\pi\rangle$ has a G-compactification.

Proof (outline). Let $\mathcal{U}^*$ be the weakest uniformity on X such that every $\mathcal{U}$-uniformly continuous function from X to the interval [0,1] is $\mathcal{U}^*$-uniformly continuous. Then $\mathcal{W}^*$ is compatible with the topology of X, and $(X,\mathcal{W}^*)$ is precompact, i.e. the completion X^* of X w.r.t. $\mathcal{W}^*$ is a compact Hausdorff space. Using (ii), it is easily seen that each $\pi^t: X \to X$ is $\mathcal{W}^*$-uniformly continuous, hence has a (uniformly) continuous extension $\sigma^t: X^* \to X^*$. Using condition (i), it is not difficult to show that the mapping $\sigma: (t,z) \mapsto \sigma^t z: G\times X^* \to X^*$ is continuous. Then $\langle G,X^*,\sigma\rangle$ is a G-compactification of $\langle G,X,\pi\rangle$. □

THEOREM 3. *Let $\langle G,X,\pi\rangle$ be a G-space with X a Tychonoff space, and suppose that there exists a uniformity $\mathcal{U}$ for X such that $\{\pi^t: t \in G\}$ is $\mathcal{U}$-equicontinuous at every point of X. Then $\langle G,X,\pi\rangle$ has a G-compactification.*

Proof (outline). For $\alpha \in \mathcal{U}$, define

$$G(\alpha) := \{(x,y) \in X \times X: (\pi^t x,\pi^t y) \in \alpha \text{ for all } t \in G\},$$

and let $\mathcal{V}$ be the uniformity, generated by $\{G(\alpha): \alpha \in \mathcal{U}\}$. Then by $\mathcal{U}$-equicontinuity, $\mathcal{V}$ is compatible with the topology of X. Moreover, $\{\pi^t: t \in G\}$ is $\mathcal{V}$-*uniformly* equicontinuous on X. Using this property of $\mathcal{V}$, it turns out that the collection of all sets of the form

$$[V,\alpha] := \{(\pi^t x,\pi^s y): s,t \in G \ \&\ ts^{-1} \in V \ \&\ (x,y) \in \alpha\},$$

V a neighbourhood of e in G and $\alpha \in \mathcal{V}$, is a base of a uniformity $\mathcal{W}$. Then $\mathcal{W}$ turns out to be compatible with the topology of X as well. The following properties of $\mathcal{W}$ and π are now easily established:

(i) $\{\pi_x: x \in X\}$ is $\mathcal{W}$-equicontinuous at e
(indeed, for every neighbourhood V of e in G and every $\alpha \in \mathcal{V}$ we have $(\pi_x t,x) = (\pi^t x,x) \in [V,\alpha]$ for all $x \in X$, provided $t \in V$);

(ii) $\forall t \in G$: $\pi^t: X \to X$ is $\mathcal{W}$-uniformly continuous
(indeed, if V is a neighbourhood of e in G and $\alpha \in \mathcal{V}$, then for every $t \in G$ there is a neighbourhood W of e in G such that $tWt^{-1} \subseteq V$, hence for all $(x,y) \in [W,\alpha]$ we have $(\pi^t x,\pi^t y) \in [V,\alpha]$). Now Theorem 2 implies that $\langle G,X,\pi\rangle$ has a G-compactification.

REMARK. In [3; 7.3.12] a different proof of Theorem 2 has been given. In fact, there we proved that condition (i) in Theorem 2 is sufficient for $\langle G,X,\pi\rangle$ to have a G-compactification $\langle G,Y,\sigma\rangle$ such that $w(Y) \leq \max\{w(G),w(X)\}$; here $w(Z)$ denotes the (topological) weight of a space Z.

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