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ON THE POINTS OF MULTIVALUEDNESS OF MONOTONE OPERATORS

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Let X be a real Banach space. The (possibly) multivalued operator $T : X \longrightarrow X^*$ defined on a domain $D(T) \subset X$ is called monotone if $\langle x_2 - x_1 , y_2 - y_1 \rangle \geq 0$ whenever $y_1 \in T(x_1)$, $y_2 \in T(x_2)$. Any subdifferential ∂f of a continuous convex function f defined on an open convex set is an example of a monotone (and also of maximal monotone) operator, but the class of all general (maximal) monotone operators is essentially larger.

There exists a number of properties of subdifferentials of continuous convex functions which have been later proved for general monotone (resp. maximal monotone) operators. It is interesting that the proofs concerning general monotone operators are frequently more simple then the original more geometrical proofs concerning convex functions. Examples:

1. Surjectivity properties of monotone operators which are proved in the theory of monotone operators.

2. Theorems concerning multivaluedness and continuity of monotone operators. Zarantonello [8] proved the first theorem of this type which generalizes the well known Mazur theorem which asserts that any convex continuous function on a separable Banach space is Gateaux differentiable instead of a set of the first category. For further theorems of this type see e.g. Aronszajn [1] , Fabián [2] , Kenderov [3] , Zajíček [6] .

3. The Mignot's [4] theorem (with a simple proof) which asserts that any monotone operator $T : R^n \longrightarrow R^n$ is almost everywhere Frechet differentiable (in a natural generalized sense). This theorem generalizes the well known Buseman-Feller-Aleksandrov theorem concerning the second differentiability of convex functions.

Thus we are led to the following

Philosophy: If any subdifferential of a continuous convex function on a Banach space X has a "regularity property" then the same property has any (maximal) monotone operator $T : X \longrightarrow X^*$.

We shall further discuss the problem of multivaluedness of monotone operators.

Notation: Let $T : X \longrightarrow X^*$ be a monotone operator. Then we denote by M_T the set of all points of multiplicity of T.

Definition: Let X be a Banach space and $0 \neq v \in X$. Let Z be a topological complement of $\mathrm{Lin}(v)$. Let f be a Lipschitz function defined on Z. Then the set $M = \{z+vf(z);\ z \in Z\}$ is termed a Lipschitz hypersurface. If f is a difference of two Lipschitz convex functions then M is termed δ-convex hypersurface.

The following theorem holds (c.f. [5] or [7]).

Theorem 1. Let X be a separable Banach space and f a continuous convex function on X. Then the set of points at which f is not Gateaux differentiable can be covered by countably many of δ-convex hypersurfaces and this is the most strong sense of smallness of this set.

In the connection with Theorem 1 and the philosophy mentioned above the following problem arises.

Problem 1. Let X be a separable Banach space and $T : X \longrightarrow X^*$ a monotone operator. Is it possible to cover M_T by countably many of δ-convex hypersurfaces ?

I know that the answer is positive if $\dim X = 2$. For general X the following less precise theorem is proved in [6].

Theorem 2. If X is separable then M_T can be covered by countably many of Lipschitz hypersurfaces.

I also do not know the solution of the following problem.

Problem 2. Let X be a Banach space and $T : X \longrightarrow X^*$ a monotone operator with open convex domain $D(T)$. Is it possible to construct a continuous convex function on D(T) which is not Gateaux differentiable in any point of M_T ?

References

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