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Δ-CLOSED GRAPH THEOREM

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Let X and Y be topological spaces. For a net $\{(x_\sigma, x'_\sigma)\}$ in $X \times X$ we write

$$(x_\sigma, x'_\sigma) \rightarrow \Delta_X$$

iff one of the following two equivalent conditions is satisfied:

(i) for any open set V containing the diagonal Δ_X there is an index σ_0 such that $(x_\sigma, x'_\sigma) \in V$ for all $\sigma \geqslant \sigma_0$;

(ii) for any open cover $\mathcal{C}$ of X there is an index σ_0 such that for every $\sigma \geqslant \sigma_0$ the set $\{x_\sigma, x'_\sigma\}$ is of diameter less than $\mathcal{C}$.

DEFINITION. A function f on X to Y has a Δ-closed graph iff

$$(x_\sigma, x'_\sigma) \rightarrow \Delta_X \text{ and } \langle f(x_\sigma), f(x'_\sigma)\rangle \rightarrow (y, y') \text{ imply } (y, y') \in \Delta_Y.$$

Proposition 1. If f is continuous and Y is a $T_{2\frac{1}{2}}$-space (= Urysohn space), then f has a Δ-closed graph. If f has a Δ-closed graph, then f has a closed graph.

Proposition 2. Suppose X and Y are topological groups. A homomorphism f from X into Y has a closed graph iff f has a Δ-closed graph.

Proposition 3. Suppose X is a $T_{2\frac{1}{2}}$-space. If the image of every closed set is closed and the counter image of any point is compact, then f has a Δ-closed graph.

A set is called nearly open iff it is in the interior of its closure. A function is called nearly continuous (nearly open) iff the counter image (image) of any open set is nearly open. A sequence $\{\mathcal{A}_n\}$ of open covers of Y is said to be complete iff any family $\mathcal{F}$ of closed subsets of Y, which has the finite intersection property and contains sets of diameter less than $\mathcal{A}_n$ for $n = 1,2,\ldots$, has non-empty intersection. $\{\mathcal{A}_n\}$ is said to be strongly countably complete iff any sequence $\{F_n\}$ of closed subsets of Y, which has the finite intersection property and consists of the sets F_n of diameter less than $\mathcal{A}_n$, has non-empty intersection. Y is said to be strongly countably complete iff there exists a strongly countably complete sequence of open covers of Y. (cf. [6]). A $T_{3\frac{1}{2}}$-space Y is Čech-complete iff there exists a complete sequence of open covers of Y (Frolík [5]). X is called a Fréchet space iff for every $A \subset X$ and every $x \in \overline{A}$ there exists a sequence $x_1, x_2, \ldots$ of points of A converging to x. (cf. [4]).
The following theorem gives a relationship between continuity, nearly continuity and Δ-closed graph property, and is the basic result of the present paper.

THEOREM. Let X and Y be topological spaces, Y a T_3-space. Suppose that

(i) Y has a complete sequence of open covers,

or

(ii) X is a Fréchet space and Y is strongly countably complete.

A function f on X to Y is continuous if and only if f is nearly continuous and has a Δ-closed graph.

The theorem implies many closed graph and open mapping theorems; we list only most general ones. Up to now, some important results concerning topological spaces and topological groups (and vector spaces) have been considered independently, using essentially different methods (see, e.g., the proofs of Corollaries 1 and 3 given in [3], [7] and [2]).

Corollary 1 (cf. [3]). Let X be a T_2-space, and let Y be a Čech-complete space. For a mapping f on X to Y the following conditions are equivalent:

(i) f is perfect;

(ii) f is nearly continuous, has a closed graph and the counter image of any compact set is compact;

(iii) f is nearly continuous, the image of any closed set is closed and the counter image of any point is compact.

Corollary 2. Let X be a Fréchet T_2-space, and let Y be a strongly countably complete T_3-space. For a mapping f on X to Y conditions (i) and (iii) of Corollary 1 are equivalent.

Corollary 3 (cf. [7] and [2]). Let X and Y be topological groups, Y Čech-complete. A homomorphism f from X into Y is continuous iff f is nearly continuous and has a closed graph.

Corollary 4. Let X be a Fréchet topological group, and let Y be a strongly countably complete T_0-group. A homomorphism f from X into Y is continuous iff f is nearly continuous and has a closed graph.

Corollary 5 (cf. [7] and [2]). Let X and Y be as in Corollary 3. A closed graph homomorphism g from Y <u>onto</u> X is open iff g is nearly open.

Corollary 6([2]). Let X and Y be as in Corollary 3. A continuous homomorphism h from Y into X is open iff h is nearly open. In other words, every Čech-complete group is B-complete.

Corollary 7 (cf. Banach [1], Pták [8] and [9]). Let X and Y be topological vector spaces, Y completely metrizable. A continuous linear mapping h from Y into X is open iff h is nearly open. In other words, every completely metrizable vector space is B-complete.

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