Jiří Vilímovský

Uniform spaces with easy behavior with respect to coreflections

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All uniform spaces are assumed to be separated, R stands for a r line, H(A) for a hedgehog over a set A (that is a cone over a uniformly discrete space A). All coreflections are assumed to be non-trivial, thus all coreflections contain all uniformly discrete spaces. We shall denote by d the coreflector onto uniformly discrete spaces, δ the coreflector onto proximally discrete spaces, t_f onto topologically fine spaces, a onto Alexandrov spaces. The last coreflection assigns to each uniform space X the coarsest uniformity finer than X and containing all finite cozero covers (see [F]). For any space X and coreflector F we shall denote X-F the class of all spaces Y such that any uniformly continuous $f:Y \longrightarrow X$ remains uniformly continuous into FX. It is well known (see [V]) that X-F is a coreflective class.

The aim of this note is to present a construction of spaces having the property that each coreflector behaves on them either identically or as d. The obtained results have some interesting consequences we want to mention shortly about. The details and proofs will appear elsewhere.

<u>Definition</u>: Let $\{k_n\}$ be a sequence of natural numbers. We define a space $D(\{k_n\})$ on a set

$$\{\langle n,i\rangle \; ; n \in N, \; 1 \leq i \leq k_n\}$$

taking $\{\mathcal{U}_m ; \; n \in N\}$, where

$$\mathcal{U}_m = \{\{\langle k,i\rangle\} \; ; \; k \leq n\} \cup \{\{\langle k,i\rangle \; ; \; i \leq k_k\} \; ; \; k > n\}$$

for a basis of uniformity.

Setting $k_n = m$, we denote the corresponding space D_N and for $k_n = 2$ we denote the space D_2.

One can easily see that spaces $D(\{k_n\})$ are complete, metrisable, zerodimensional and topologically discrete.

Proposition 1: Let $\mathcal{E}$ be a coreflective subcategory of uniform spaces, F the corresponding coreflector, $\{k_n\}$ a sequence of natural numbers. Suppose $D(\{k_n\}) \notin \mathcal{E}$, then $FD(\{k_n\})$ has a discrete proximity. (All pairs of disjoint sets are proximally far).

That means that taking any sequence $\{k_n\}$, then either $FD(\{k_n\}) = D(\{k_n\})$ or $FD(\{k_n\})$ is finer than $\delta D(\{k_n\}) = aD(\ k_n\ )$. Moreover if $\{k_n\}$ is bounded, then either $FD(\{k_n\}) = D(\{k_n\})$ or $FD(\{k_n\})$ is uniformly discrete. We obtain the following

Corollary 1: The following properties of a uniform space X are equivalent:

(1) X is $D(\{k_n\}) - \delta$ for all sequences $\{k_n\}$.

(2) X is $D(\{k_n\}) - d$ for all bounded sequences $\{k_n\}$.

(3) X is $D_N - \delta$

(4) X is $D_2 - d$

Having any coreflective class $\mathcal{E}$ in uniform spaces, the obtained result gives that either $\mathcal{E}$ is contained in D_2-d, or $\mathcal{E}$ containes the coreflective hull coref(D_2) of $\{D_2\}$. One may find interesting that coref(D_2) is very "large", in fact it containes all metrisable spaces, what follows from the following easy statement (cf. [Č]):

For M,S metrisable, $f:M \longrightarrow S$ is uniformly continuous if and only if fg is uniformly continuous for all $g:D_2 \longrightarrow M$ uniformly continuous.

One may go a step further from the Proposition 1 proving:

Proposition 2: Take any coreflector F in uniform spaces, then either $FD_N = D_N$ or $FD_N = \delta D_N$ or $FD_N = dD_N$.

Instead of D_N we can take any space $D(\{k_n\})$ for an unbounded sequence $\{k_n\}$ of natural numbers. Again similar conclusions as for D_2

Corollary 2: Let $\mathcal{E}$ be any coreflective class in uniform spaces and let neither D_N nor δD_N be in $\mathcal{E}$. Then $\mathcal{E}$ is a subclass of D_N-d.

More interesting results can be obtained if we restrict ourselves to coreflective classes closed under subspaces. We recall that for any coreflective class $\mathcal{E}$ the class Sub($\mathcal{E}$) of all subspaces of spaces in $\mathcal{E}$ forms again a coreflective class (see [V]). A similar theorem for the class Her($\mathcal{E}$) of spaces being hereditarily in $\mathcal{E}$ is not valid in general, but fortunately in the case of D_2-d we obtain again a coreflection having even a nice description:

Theorem 1: The following properties of a uniform space X are equivalent:

(1) X is hereditarily D_2-d

(2) X is hereditarily $D(\{k_n\})$ - d for all bounded $\{k_n\}$.

(3) X is hereditarily $D(\{k_n\})$ - δ for all $\{k_n\}$.

(4) Xis hereditarily $D_N - \delta$

(5) Each countable uniformly discrete union of boundedly finite uniformly discrete families is uniformly discrete.

(6) X is $H(\omega)$-e

(7) X is hereditarily R-e

(8) For any countable family $\{f_n\}$ of uniformly bounded and uniformly continuous real valued functions on X with $\{\text{supp } f_n\}$ uniformly discrete, the function $\sum' f_n$ is uniformly continuous.

(9) For every $Y \subset X$, $f: Y \to R$ uniformly continuous, $g: R \to R$ continuous bounded, the function gf is uniformly continuous.

Spaces being hereditarily D_N-d have again very nice properties and form a coreflective class. These spaces are studied in [FPV] . We recall at least some most interesting properties of them:

Theorem 2: The following properties of a uniform space X are equivalent

(1) X is hereditarily D_N-d

(2) X is $H(\omega) - t_f$

(3) For every $Y \hookrightarrow X$, the set $U(Y)$ of all uniformly continuous real valued functions is a ring.

(4) For any sequence $f_n \in U(X)$ such that f_n are bounded and the family $\{\text{supp } f_n\}$ is uniformly discrete, the sum $\sum f_n$ is uniformly continuous.

(5) $U(X)$ is a ring and for any $Y \hookrightarrow X$, $f \in U(Y)$, there exists an extension $\bar{f} \in U(X)$ of f.

We shall denote these two coreflections $H(\omega) - a$ and $H(\omega) -$ respectively.

In order to make some conclusions from the remark after Proposition 1, we must know, what is $\text{Sub}(\text{coref}(D_2))$. It is clear that it is a very large coreflective class containing all metric spaces. Under some set theoretic assumptions, $\text{Sub}(\text{coref}(D_2))$ may be even the class of all uniform spaces. Assuming [SEQ], the nonexistence of Mazur´s sequential cardinals, then $\text{coref}(D_2)$ is productive (see [H]), hence $\text{Sub}(\text{coref}(D_2))$ contains all uniform spaces. Thus under this assumption we have:

<u>Theorem 3</u>: [SEQ] The class $H(\omega) - a$ is the largest nontrivial hereditary coreflective subcategory of uniform spaces.

Further application of our construction may be the following, suggested by Corollary 2: Having any class $\mathcal{A}$ of uniform spaces closed under subspaces. If neither D_2 nor δD_N are in $\mathcal{A}$, then whenever $H(\omega)-t_f \subset \mathcal{A}$, then $H(\omega)-t_f$ is the largest coreflectiv class contained in $\mathcal{A}$. For example we can prove the following:

<u>Theorem 4</u>: $H(\omega)-t_f$ is the largest coreflective class contained in the following classes:

(a) The class of all X such that for any subspace Y of X, $f \in U(Y)$, there is an extension $\bar{f} \in U(X)$ of f.

(b) The class of all X such that for every free uniform measure μ on X the support supp($\check{\mu}$) of the corresponding Radon measure on the Samuel compactification $\check{X}$ of X lies in the completion $\hat{X}$ of X.

(c) The class of all spaces with the property that each bounded subset of it is precompact.

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