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A PARTITION OF $\mathbb{R}$ IN TWO HOMOGENEOUS AND HOMEOMORPHIC PARTS

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1. INTRODUCTION.

In this paper we give a sketch of the construction of the partition.

Consider the maps $P(x) = 3^n x + 2k.3^m$; $n,k,m \in \mathbb{Z}$, and denote by $P = \{P_n | n \in \mathbb{N}\}$ the set of these. In paragraph 2, it is proved that if $A \subset \mathbb{R}$ is stable for P, and $1 \in A$, then A is necessarily homogeneous. We then consider the sets X(A), the orbits of a by P. In paragraph 4 it is proved that if a function f with certain properties exists, the required partition can be constructed. In paragraph 5 a sketch is given of the construction of such a function, omitting the technical parts.

I am indebted to Prof. Maurice to have suggested the problem and for valuable discussions.

2. PROPOSITION. Let $A \subset \mathbb{R}$ be stable for the maps P_n, $1 \in A$, then A is homogeneous.

PROOF. Let $x \in A$, it is sufficient to construct a homeomorphism $h : A \to A$, such that $h(x) = 1$

a) apply a translation $h_0 : A \to A,\ x \to x + 2k.3^m$ such that $h(x_0) \in]0,2[$

b) 1) if $h_0(x) \in]\frac{2}{3},\frac{4}{3}[$, define $h_1 = h_0$

2) if $h_0(x) < \frac{2}{3}$, then there exists just one $n_1 \in \mathbb{Z}$ such that

$2.3^{n_1} < h_0(x) < 2.3^{n_1-1}$

$$\text{define } h_1(y) = \begin{cases} h_0(y) & \text{if } h_0(y) \notin]0,2[\\ 3^{n_1+1}.h_0(y) & \text{if } 0 < h_0(y) < 2.3^{n_1} \\ h_0(y) + (\frac{2}{3} - 2.3^{n_1}) & \text{if } 2.3^{n_1} < h_0(y) < \frac{4}{3} \\ 3^{-n_1-1}(h_0(y)-\frac{4}{3}) + (2-2.3^{n_1}) & \text{if } \frac{4}{3} < h_0(y) < 2 \end{cases}$$

$\Rightarrow h_1(x) \in]\frac{2}{3},\frac{4}{3}[$

3) if $h_0(x) > \frac{4}{3}$, construct h_1 in an analogous way.

c) Suppose h_n is constructed, $h_n(x) \in]1-3^{-n},1+3^{-n}[$

1) if $h_n(x) \in]1-3^{-n-1},1+3^{-n-1}[$

define $h_{n+1} = h_n$

2) if $h_n(x) \in]1-3^{-n},1-3^{-n-1}[$

then there exists just one $n_{n+1} \in \mathbb{Z}$ such that

$1-3^{-n} + 2.3^{-n+n_{n+1}} < h_n(x) < 1-3^{-n} + 2.3^{n_{n+1}-n+1}$

$$h_{n+1}(y) = \begin{cases} h_n(y) & \text{if } h_n(y) \notin]1-3^{-n},1+3^{-n}[\\ 1-3^{-n} + 3^{-n_{n+1}-1}(h_n(y)-(1-3^{-n})) & \\ \quad \text{if } 1-3^{-n} < h_n(y) < 1-3^{-n} + 2.3^{n_{n+1}-n} & \\ h_n(y) + 2.3^{-n-1} - 2.3^{n_{n+1}-n} & \\ \quad \text{if } 1-3^{-n} + 2.3^{n_{n+1}-n} < h_n(y) < 1+3^{-n-1} & \\ 3^{n_{n+1}+1}(h_n(y)-(1+3^{-n-1})) + 1+3^{-n} - 2.3^{n_{n+1}-n} & \\ \quad \text{if } 1+3^{-n-1} < h_n(y) < 1+3^{-n} & \end{cases}$$

$\Rightarrow h_{n+1}(y) \in]1-3^{-n-1},1+3^{-n-1}[$

3) if $h_n(x) \in \,]1+3^{-n-1}, 1+3^{-n}[$, construct h_{n+1} in an analogous way.

d) define $h(y) = \lim_n h_n(y)$.

It is easy to see that h is a homeomorphism of A and $h(x) = 1$.

3. The classes $X(a)$, $a \in \mathbb{R}$.

1. Define $X(a) = \{3^m a + 2k.3^n \mid m,n,k \in \mathbb{Z}\}$. Every $X(a)$ is countable, and the set $\{X(a) \mid a \in \mathbb{R}\}$ is a partition of $\mathbb{R}$.

2. $$a + 1 \in X(a) \Leftrightarrow a + 1 = 3^n a + 2k.3^m \qquad (n,k,m \in \mathbb{Z})$$

$$\Leftrightarrow a(3^n - 1) = 1 - 2k.3^m$$

$$\Leftrightarrow a = \frac{1-2k.3^m}{3^n-1}$$

$$\Leftrightarrow a \in X\left(\frac{2l+1}{3^p-1}\right) \quad \text{for some } p,l \in \mathbb{N} \text{ and } 0 < 2l+1 < 2.3^p .$$

In this case $X(a+1) \cap X(a) \neq \phi$, and $X(a+1) = X(a)$. Denote by $(K_n)_n$ these classes.

3. Let $a \in X\left(\frac{2l+1}{3^n-1}\right) = K_q$ where $l,n \in \mathbb{N}$, $0 < 2l+1 < 2.3^n$ and such that $\forall\, k',n' \in \mathbb{N};\ n' < n$:

$$\frac{2k'+1}{3^{n'}-1} \notin X\left(\frac{2l+1}{3^n-1}\right) .$$

then there are unique k,p,m with $0 \leq m < n$, such that

$$a = 3^m.\frac{2l+1}{3^n-1} + \frac{k}{3^p}.$$

Define $D(q,k) = \left\{x \mid x = 3^m.\frac{2l+1}{3^n-1} + \frac{r}{3^p},\ \begin{array}{l} 0 \leq p \leq k \\ 0 \leq m < n \\ r \in \mathbb{Z} \end{array}\right\}$ where $K_q = X\left(\frac{2l+1}{3^n-1}\right)$.

4. $R_1 = \mathbb{R} \setminus X(0)$.

4. <u>THEOREM</u>. Suppose there is a continuous function $f : R_1 \to R_1$, such that :

$$\begin{cases} 1)\ f^2 = 1_{R_1} \\ 2)\ \forall x : f(x) \notin X(x+1) \\ 3)\ \forall x : X(f(x)) = f(X(x)) \\ 4)\ \text{if } g(x) = f(x) + 1 \text{ then :} \\ \quad \forall x \notin X(1), \forall n \in \mathbb{N} : x \notin X(g^{2n+1}(x)) \end{cases}$$

$$5)\ f(1) \in X(1)$$

then there is a partition of $\mathbb{R}$ in two homeomorphic parts.

<u>Remark</u>. $g(X(x)) = X(g(x)), \quad \forall x \in R_1$.

<u>PROOF of the theorem.</u>

1. $\forall x \in R_1 \setminus X(1)$; define

$$A_x = \cup\{X(g^{2n}(x)) | n \in \mathbb{Z}\}$$

$$B_x = \cup\{X(g^{2n+1}(x)) | n \in \mathbb{Z}\}.$$

Suppose $g^{2n}(x) \in X(g^{2m+1}(x))$, then $x \in g^{-2n}(X(g^{2m+1}(x))) = X(g^{2m-2n+$ which contradicts (4), thus $A_x \cap B_x = \phi$.

2. Choose $x_1 \notin X(1) \cup X(0)$. Define $A_1 = A_{x_1}$, $B_1 = B_{x_1}$

$$A_1 \cap B_1 = \phi.$$

3. Suppose A_β and B_β are defined for every ordinal $\beta < \alpha$, $A_\beta \cap B_\beta = \phi$. Then a) if α is a limit-ordinal, define

$$\begin{cases} A_\alpha = \cup\{A_\beta | \beta < \alpha\} \\ B_\alpha = \cup\{B_\beta | \beta < \alpha\} \end{cases}$$

b) if α is not a limit-ordinal,

i) $A_\beta \cup B_\beta \cup X(0) \cup X(1) = \mathbb{R}$, for some $\beta < \alpha$ define

$$A_\alpha = \cup\{A_\beta | \beta < \alpha\}$$

$$B_\beta = \cup\{B_\beta | \beta < \alpha\}$$

ii) $A_{\alpha-1} \cup B_{\alpha-1} \cup X(0) \cup X(1) = C_{\alpha-1} \neq \mathbb{R}$. Choose $x_\alpha \in \mathbb{R} \setminus C_{\alpha-1}$.

Define $\begin{cases} A_\alpha = A_{\alpha-1} \cup A_{x_\alpha} \\ B_\alpha = B_{\alpha-1} \cup B_{x_\alpha} \end{cases}$

for every α such that $A_\alpha \cap B_\alpha = \phi$ and $g(A_\alpha) = B_\alpha$. There exists an ordinal α_0 such that $A_{\alpha_0} \cup B_{\alpha_0} \cup X(1) \cup X(0) = \mathbb{R}$.

4. Define $\begin{cases} A = \cup\{A_\alpha | \alpha \leq \alpha_0\} \cup X(1) \\ B = \cup\{B_\alpha | \alpha \leq \alpha_0\} \cup X(0). \end{cases}$

Then : $\begin{cases} A \cap B = \phi \\ A \cup B = \mathbb{R} \end{cases}$

$g : A \to B$ is a homeomorphism.

4.2. This proves that $\mathbb{R} = A \cup B$, where

$$\begin{cases} A \cap B = \phi \\ A \cong B \\ A \text{ and } B \text{ homogeneous.} \end{cases}$$

5. Sketch of the construction of f.

5.1. a) We only consider intervals $[a,b]$ with $a,b \in X(0)$, and $b-a = 2.3^k$ for some $k \in \mathbb{Z}$.

A net N on such an interval is a finite set of points $a = a_0 < a_1 <\ldots< a_{n-1} < a_n = b$, such that $\forall\, i \leq n-1$, there exists a $m \in \mathbb{Z} : a_{i+1} - a_i = 2.3^m$:

b) Let N be a net on $[0,2]$, $x \in [0,2]$. Denote :

$V_N(x) = \cup\{V | V \in N, x \in \overline{V}\}$.

c) Let $[a,b]$ be an interval, N a net on $[a,b]$. A function

$$\begin{cases} 1)\ \forall\, i < n-1 : h|_{]a_i,a_{i+1}[} \text{ is a translation onto} \\ \quad \text{some }]a_j,a_{j+1}[,\ j < n-1 \\ 2)\ h \text{ is the identity on }]a_0,a_1[,\ ]a_{n-1},a_n[\text{ and on} \\ \quad \text{the interval containing } \frac{a+b}{2} \\ 3)\ h^2 = 1|_{[a,b] \cap R_1} \\ 4)\ \forall\, x \in R_1 : h(x) + 2 = h(x+2) \end{cases}$$

5.2. At the n^{th} step is constructed the following :

1) A finite set $B_n \subset \cup\{K_m | m \in \mathbb{N}\}$, such that $B_{n-1} \subset B_n$ and $\cup\{B_m | m \in \mathbb{N}\} = \cup\{K_m | m \in \mathbb{N}\}$.

2) $\forall\, k < n$ and $x,y \in B_n \cap K_k$ a $P^k_{x,y} \in \{P_m | m \in \mathbb{N}\}$, such that $P^k_{x,y}(x) = y$, and if $\mathcal{P}_n\{P^k_{x,y} | x,y \in B_n \cap K_k,\ k < n\}$, then $\mathcal{P}_{n-1} \subset \mathcal{P}_n$ and $\mathcal{P}_n$ is a transitive representation group.

3) $\forall\, x \in B_{n-1} \cap K_k$, $k < n-1$, a $V^k_{x,n-1} \in \mathcal{V}(x)$, such that $P^k_{x,y}(V^k_{x,n-1}) = V^k_{y,n-1}$ and if $k' < n$, $1 < n-1$, $V^k_{x,n-1} \cap$ $\cap\, V^k_{y,1} \neq \phi \Rightarrow V^k_{x,n-1} \subset V^{k'}_{y,1}$. Denote $V_n = \cup\{V^k_{x,n-1} | x \in B_{n-1} \cap K_k, k < n-1\}$

4) A net $\mathcal{N}_{n-1}$ that refines $\mathcal{N}_{n-2}$ and contains the endpoints of the $V^k_{x,n-1}$, $k < n-1$, $x \in B_{n-1} \cap K_k$, and such that

$$P^k_{x,y}(\mathcal{N}_{n-1}|_{V^k_{x,n-1}}) = \mathcal{N}_{n-1}|_{V^k_{y,n-1}} .$$

5) A function f_{n-1} such that

a) $f_{n-1}|_{[0,2] \cap (R_1 \setminus F_{n-2}(V_{n-1}))} = 1$, where

$F_{n-2} = f_{n-2} \circ f_{n-3} \circ \ldots \circ f_1$.

b) f_{n-2} is a $\mathcal{N}_2$-elementary function on every $F_{n-2}(V^k_{x,n-1})$

c) $\forall\, x,y : P^k_{x,y} \circ f_{n-1} \circ F_{n-2}(x') = f_{n-1} \circ F_{n-2} \circ P^k_{x,y}(x')$,

$x' \in V^k_{x,n-1}$.

6) Moreover, there is taken care of the following :

a) $x \in K_{2m-1} \Rightarrow \lim_n F_n(x) \in K_{2m}$

b) the condition 4) is satisfied in the n^{th} step outside of a set of length $< 3^{-n}$.

Define $f = \lim_n F_n$, then clearly f has the required properties, and this completes the proof.