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EXTENSION OF FUNCTIONS ON BOOLEAN ALGEBRAS

BY

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Let A be a Boolean algebra with the operations of join, meet and difference denoted by $\vee$, $\wedge$ and $\setminus$, respectively. The natural ordering of A is denoted by $\leq$ and its minimal element by 0.

Given a subalgebra B of A and a function μ on B with values in $[0, \infty)$ or in an Abelian (topological) group G which has some properties related to the structures of both the domain and range spaces, we consider the problem of extending μ to A all the properties preserved.

I. Suppose $\mu: B \to [0, \infty)$. We are concerned with four sets of conditions imposed on μ.

(i) μ is monotone (i.e. $\mu(b) \leq \mu(c)$ whenever $b, c \in B$ and $b \leq c$), subadditive (i.e. $\mu(b_1 \vee b_2) \leq \mu(b_1) + \mu(b_2)$ whenever $b_1, b_2 \in B$) and $\mu(0) = 0$ (¹).

(ii) μ is additive (i.e. $\mu(b_1 \vee b_2) = \mu(b_1) + \mu(b_2)$ whenever $b_1, b_2 \in B$ and $b_1 \wedge b_2 = 0$).

(iii) μ is monotone, subadditive and exhaustive (i.e. $\mu(b_i) \to 0$ whenever $\{b_i\} \subset B$ and $b_i \wedge b_{i'} = 0$ for all $i \neq i'$).

(iv) μ is monotone, exhaustive and $\mu(b \vee c) = \mu(b \setminus c)$ $(= \mu(b))$ whenever $b, c \in B$ and $\mu(c) = 0$.

(¹) Such a function is usually called a submeasure.

As easily seen, (ii)$\Rightarrow$(i) and (ii)$\Rightarrow$(iii)$\Rightarrow$(iv) and none of these implications can be reversed.

The case where μ satisfies (i) is trivial. Putting $\nu(a) = \inf\{\mu(b):\ a \leq b \in B\}$ for $a \in A$, we get an extension satisfying (i).

The answer is also positive in case μ fulfils (ii) or (iii). As for (ii), this is a classical result of Hahn-Banach type (see, e.g., [4], p. 270). When dealing with (iii), the transfinite argument must be essentially improved since the exhaustivity condition involves countably many elements of A. The improvement is suggested by the following observations:

1. For any μ satisfying (i) the formula $d_\mu(b_1, b_2) = \mu((b_1 \setminus b_2) \vee (b_2 \setminus b_1))$ for $b_1, b_2 \in B$ defines a pseudometric on B.

2. If $C \subset B$ is a d_μ-dense subalgebra of B and $\mu|C$ is exhaustive, then so is μ.

(The details can be reconstructed from [3]. For a different proof see [2].)

As for (iv), the answer is positive under the additional assumption that B has the following compactness type property investigated by Seever [5]:

(I) Suppose $b_n, c_n \in B$ and $\bigvee_{i=1}^{k} b_i \leq \bigvee_{i=1}^{k} c_i$ for $k = 1, 2, \ldots$ Then there exists $x \in B$ with $\bigvee_{i=1}^{k} b_i \leq x \leq \bigvee_{i=1}^{k} c_i$ for $k = 1, 2, \ldots$

This property, which is somewhat weaker than σ-completeness, corresponds to the fact that the representation space of B is an F-space ([5], Theorem A). The author does not know whether the result fails for B without (I) (²). Before sketching

(²) For an arbitrary B it can be shown that if μ fulfils (iv) with "exhaustive" weakened to "$\mu(0) = 0$", then it extends to A all its properties preserved.

the proof we give a lemma which is a slight generalization of a result due to Seever ([5], Lemma 3.3).

Lemma. If B has the property (I), N is an ideal in B and B/N satisfies the countable chain condition (CCC), then B/N is complete.

Suppose μ satisfies (iv) and B has the property (I). Put $N = \{b \in B: \mu(b) = 0\}$. Then B/N satisfies CCC, so that, by the Lemma and Sikorski's theorem ([6], 33.1), the quotient homomorphism $B \to B/N$ extends to a homomorphism $h: A \to B/N$. Putting $\nu(a) = \tilde{\mu}(h(a))$, where $\tilde{\mu}: B/N \to [0, \infty)$ is the quotient of μ, we get the desired extension (cf. also [2], Theorem 1).

II. Suppose G is an Abelian complete Hausdorff topological group and $\mu: B \to G$ is additive and exhaustive. (These properties are defined in just the same way as in I.) Then μ extends to A both properties preserved ([3], Theorem 3, and [2], Corollary 3). This theorem can be proved using a similar idea to that described when dealing with μ satisfying (iii). In fact, it is not hard to see that the group-valued case is more general than (iii). Let us also note that the assumption that G is complete cannot be dropped ([3], Example 4).

III. Suppose G is an Abelian group and $\mu: B \to G$ is additive. In this case the extension problem is open in general (e.g. for G being the additive group of integers). The answer is positive under each of the following additional assumptions:

(a) (W. Herer) G is injective (see [1], §21, for definition).

(b) (C. Ryll-Nardzewski) card $A \leq \aleph_1$.

(c) B/N is complete, where $N = \{b \in B: \mu(c) = 0$ for all $b \geq c \in B\}$.

As for (c), the assertion follows by an application of Sikorski's theorem analogously to the case where μ satisfies (iv).

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