

R. Huff

Asplund spaces and the RNP

In: Zdeněk Frolík (ed.): Abstracta. 5th Winter School on Abstract Analysis. Czechoslovak Academy of Sciences, Praha, 1977. pp. 29--30.

Persistent URL: <http://dml.cz/dmlcz/701083>

## Terms of use:

© Institute of Mathematics of the Academy of Sciences of the Czech Republic, 1977

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* <http://project.dml.cz>

## FIFTH WINTER SCHOOL (1977)

## Asplund Spaces and the RNP

R. Huff

A real Banach space  $X$  is an Asplund space provided for every open convex set  $U \subset X$ , every continuous convex functional  $\phi : U \rightarrow \mathbb{R}$  is (Frechet-) differentiable on a dense  $G_\delta$  subset of  $U$  [3]. The main result discussed is the following.

THEOREM (Asplund, Namioka, Phelps, Stegall). The following two statements are equivalent:

- (1)  $X$  is an Asplund space.
- (2)  $X^*$  has the Radon-Nikodym property (RNP).

A proof was given by breaking the result into two parts, the first of which is

THEOREM ([3], [4]). Statement (2) above is equivalent to

- (3) For every closed bounded set  $A \subset X^*$ , the set

$$\{x \in X : \tilde{x} \text{ strongly exposes } A\}$$

is a dense  $G_\delta$  subset of  $X$ .

The second part, of course, is to show that (3) and (1) are equivalent; a proof can be found in [3], but here we gave a new proof via the following two easy lemmas.

DUALITY LEMMA. Let  $A$  be any closed bounded subset of  $X^* \times \mathbb{R}$ , and let  
 $\varphi(x) = \sup\{f(x)+\alpha : (f,\alpha) \in A\}$  ( $x \in X$ ). Then  $\varphi$  is a convex functional on  
 $X$  satisfying a Lipschitz condition. Moreover,  $\varphi$  is differentiable at  $x$   
(with gradient  $g$ ) iff the map on  $X^* \times \mathbb{R}$  sending  $(f,\alpha)$  into  $f(x)+\alpha$  strongly  
exposes  $A$  (at the point  $(g,\varphi(x)-g(x))$ ).

REDUCTION LEMMA. Let  $U$  be any open convex subset of  $X$  and  $\varphi : U \rightarrow \mathbb{R}$   
any continuous convex functional. For each  $n = 1, 2, \dots$ , let

$$A_n = \{(f,\alpha) \in X^* \times \mathbb{R} : f(x)+\alpha \leq \varphi(x), \text{ all } x \in U, \|f\| \leq n, |\alpha| \leq n\},$$

and let  $\varphi_n(x) = \sup\{f(x)+\alpha : (f,\alpha) \in A_n\}$ . For every  $x$  in  $U$  there exist  
 $\delta > 0$  and  $N$  such that

$$\|y-x\| < \delta \Rightarrow y \in U \quad \text{and} \quad \varphi(y) = \varphi_n(y) \quad \text{for all} \quad n \geq N.$$

#### References

1. E. Asplund, Frechet differentiability of convex functions, Acta Math. 121  
(1968), 31-47
2. R. Huff, preprint.
3. I. Namioka and R.R. Phelps, Banach spaces which are Asplund spaces,  
Duke Math. J. 42(1975).
4. C. Stegall, in preparation.