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TWO SET-THEORETIC PROBLEMS IN TOPOLOGY

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The aim of this lecture is to draw your attention to some particular problems, rather than giving a survey of an area. I believe that the eventual solutions of these problems will significantly contribute to the progress of set-theoretic topology.

§.1. ON THE NUMBER OF OPEN SETS

For any topological space X I denote by $o(X)$ the number of all open subsets of X, i.e. the cardinality of the topology of X. J. de Groot raised the following problem in [3] : If X is an infinite Hausdorff space is $o(X)$ necessarily of the form $2^{\varkappa}$? He has observed that this is so for metric spaces. In [5] it has been shown that the answer to de Groot's question is affirmative if GCH and the non-existence of inaccessible cardinals are assumed.

On the other hand it follows from results in [6],[7] and [8] that in some models of set theory there are very good topological spaces such that e.g. $2^{\omega} < o(X) < 2^{\omega_1}$. In fact these spaces can be chosen as hereditarily separable topological groups or as regular and hereditarily Lindelöf. These results leave open the following problem:

1.1. PROBLEM. Let X be an infinite T_2 space. Is $o(X)^{\omega}=o(X)$?

The naturalness of this question is accentuated by a known result of R.S. Pierce and B. Efimov (cf. [2] and [15]) saying that the cardinality $\varkappa$ of an infinite complete Boolean algebra (i.e. the number of all regular open sets in a space) always satisfies $\varkappa^{\omega}=\varkappa$.

In what follows we present several partial results concerning problem 1.1. We shall be frequently using the following results on cardinal exponentiation (see e.g. [11]):

1.2. PROPOSITION. Let $\varkappa$ and λ be infinite cardinals. Then

A) $(\varkappa^{+})^{\lambda}=\varkappa^{+}\cdot\varkappa^{\lambda}$;

B) $\lambda < cf(\varkappa)$ implies $\varkappa^{\lambda}=\sum\{\alpha^{\lambda}:\alpha < \varkappa\}$;

C) if $\lambda=cf(\varkappa)$ and $\varkappa=\sum\{\varkappa_\nu:\nu\in\lambda\}$, where $\varkappa_\nu < \varkappa$ for $\nu\in\lambda$, then

$$\varkappa^\lambda = \Pi\{\varkappa_\nu : \nu\in\lambda\};$$

D) if $cf(\varkappa) < \lambda < \varkappa$, then

$$\varkappa^\lambda = (\sum\{\alpha^\lambda : \alpha < \varkappa\})^{cf(\varkappa)}$$

The next result also concerns cardinal exponentiation, and as far as I know it is new. It will play a crucial role in the proof given below.

Let $\varkappa$ and λ be cardinals, the power $\varkappa^\lambda$ is called a *jump*, if $\varkappa,\lambda \geq \omega$, $\alpha < \varkappa$ implies $\alpha^\lambda < \varkappa^\lambda$ and $\beta < \lambda$ implies $\varkappa^\beta < \varkappa^\lambda$.

1.3. LEMMA. If $\varkappa^\lambda$ is a jump, then $\lambda=cf(\varkappa)$.

Proof. First we show that $\alpha < \varkappa$ implies $\alpha^\lambda < \varkappa$. Indeed $\alpha^\lambda \geq \varkappa$ would imply $(\alpha^\lambda)^\lambda=\alpha^\lambda\geq\varkappa^\lambda$, a contradiction. In particular we obtain $2^\lambda < \varkappa$, hence $\lambda < \varkappa$.

Now assume that $\lambda < cf(\varkappa)$. Then by 1.2 *B)* and the above we have

$$\varkappa^\lambda = \sum\{\alpha^\lambda: \alpha < \varkappa\} \leq \varkappa=\varkappa^1 < \varkappa^\lambda$$

again a contradiction.

Finally, $cf(\varkappa) < \lambda$ would imply by 1.2 *D)* that

$$\varkappa^\lambda = (\sum\{\alpha^\lambda: \alpha < \varkappa\}^{cf(\varkappa)} \leq \varkappa^{cf(\varkappa)} < \varkappa^\lambda,$$

which is impossible. Thus, indeed, we must have $\lambda=cf(\varkappa)$.

The following three simple statements will be often used without mention in what follows. Their proofs are left to the reader.

1.4. LEMMA. Let R be an arbitrary topological space.

1) If $R=\cup\{R_i: i\in I\}$, then $o(R) \leq \Pi\{o(R_i): i\in I\}$.

2) If $\{R_i: i\in I\}$ is a disjoint family of non-empty open subspaces of R, then

$$o(R) \geq \Pi\{o(R_i): i\in I\}.$$

3) If there is a discrete subspace of cardinality $\varkappa$ in R, then $o(R) \geq 2^\varkappa$.

Let us denote by $\mathcal{H}$ the class of all strongly Hausdorff spaces (cf. [13]). Our next result, as we shall see, makes it very probable that the answer to problem 1.1 is affirmative at least for spaces in $\mathcal{H}$.

1.5. THEOREM. Let $\varkappa$ be a cardinal such that $o(X)=\varkappa$ for some infinite $X\in\mathcal{H}$ and $\varkappa < \varkappa^{\omega}$. Then there is a cardinal β with the following properties *(i) - (iv)*:

(i) $\omega < cf(\beta) = \gamma < \beta$;

(ii) $(\forall\alpha < \beta)(\alpha^{\gamma} < \beta)$;

(iii) $\beta^{\gamma} > \beta^{(\omega)}$ (=the ω^{th} successor of β);

(iv) $\varkappa \geq \beta^{(\omega)}$.

Proof. Let λ be the smallest cardinal such that $\lambda^{\omega} > \varkappa$. Since $\lambda \leq \varkappa$, the power λ^{ω} is clearly a jump, hence by 1.3 we have $\omega=cf(\lambda)$. Moreover $\varkappa=o(X) > 2^{\omega}$ implies $\lambda > \omega$.

For any $p\in X$ let us put

$$\sigma(p,X) = \min\{o(U) : p\in U, \quad U \text{ open in } X\}$$

and

$$\sigma = \sigma(X) = \sup\{\sigma(p,X) : p\in X\}.$$

Since $X\in\mathcal{H}$ there can only be finitely many points $p\in X$ such that $\sigma(p,X) \geq \lambda$, for otherwise X would contain a disjoint family $\{U_n : n\in\omega\}$ of open sets with $o(U_n) \geq \lambda$ for all $n\in\omega$, and thus

$$o(X) \geq \lambda^{\omega} > \varkappa = o(X)$$

would follow. On the other hand, throwing away finitely many points from X will clearly not change $o(X)$, hence we can assume that $\sigma(p,X) < \lambda$ for each $p\in X$.

Now we claim that in fact $\sigma < \lambda$ must be valid. Assume, on the contrary, that $\sigma=\lambda$. Since λ can be written as $\lambda=\sum\{\lambda_n : n\in\omega\}$, where $\lambda_n < \lambda$ for $n\in\omega$, then we can pick for $n\in\omega$ distinct points $p_n\in X$ such that $\sigma(p_n,X) > \lambda_n$, moreover using $X\in\mathcal{H}$ we can also assume that each p_n has a neighbourhood U_n so that the family $\{U_n : n\in\omega\}$ is disjoint. However this implies, by 1.2 *c)*,

$$o(X) \geq \Pi\{o(U_n) : n\in\omega\} \geq \prod_{n\in\omega} \lambda_n = \lambda^{\omega} > \varkappa,$$

a contradiction.

Next we show that $|X| \leq \sigma^+$. Indeed, every $p \in X$ has an open neighbourhood $U(p)$ such that $|U(p)| \leq o(U(p)) \leq \sigma$. Hence if $|X| > \sigma^+$ were true then $U(p)$ would be a set-mapping which satisfies the conditions of Hajnal's theorem (cf [4] or [13]), hence a free set $D \subset X$ with $|D| = |X|$ would exist for $U(p)$. However this subspace D is clearly discrete, consequently $\varkappa = o(X) = 2^{|X|}$, which is of course impossible.

Now consider the above defined open cover $\mathcal{U} = \{U(p) : p \in X\}$ of X, then $|\mathcal{U}| \leq \sigma^+$. Let τ denote the smallest cardinal for which X *does not* contain a discrete subspace of cardinality τ. As is shown in [9], then every closed subset $F \subset X$ can be obtained in the following form

$$F = (F \cap (\cup \mathcal{U}_F)) \cup \bar{S}_F ,$$

where $\mathcal{U}_F \in [\mathcal{U}]^{<\tau}$ and $S_F \in [X]^{<\tau}$. An easy calculation shows then that

$$\lambda \leq o(X) \leq (\sigma^+)^{\underset{\smile}{\tau}}.$$

Since $X \in \mathcal{H}$, then 3.3 of [13] implies $cf(\tau) > \omega$. From this and $cf(\lambda) = \omega$ it follows then that there is a cardinal $\rho < \tau$ with $(\sigma^+)^{\rho} > \lambda$. Let γ be the smallest cardinal with $(\sigma^+)^{\gamma} > \lambda$ and then β be smallest such that $\beta^{\gamma} > \lambda$. Then $\beta \leq \sigma^+ < \lambda$, hence $\gamma > \omega$ by the choice of λ. Moreover $\gamma < \tau$, hence X contains a discrete subspace of size γ, consequently $o(X) \geq 2^{\gamma} = \gamma^{\gamma}$ and thus $\beta^{\gamma} \geq \lambda^{\omega} > o(X)$ implies $\beta > \gamma$. In particular β and γ are infinite, hence the power β^{γ} is a jump and therefore $\gamma = cf(\beta)$. Now it is obvious that β satisfies conditions *(i) - (iv)*.

As an immediate corollary we obtain the main result of [9] saying that if $X \in \mathcal{H}$ and $o(X) < \omega_{\omega_1 + \omega}$ then $o(X)^{\omega} = o(X)$. Indeed, this is obvious since ω_{ω_1} is the smallest cardinal which satisfies *(i)*. However our result says much more than this. Indeed, the consistency of the existence of a cardinal satisfying *(i) - (iii)* has only been established by M. Magidor [14] with the help of some enormously large (so called strongly compact) cardinals. Moreover by some very recent results of R. Jensen [12] the existence of such a β implies that measurable cardinals exist in some inner models of set theory. This shows that constructing a "counterexample" to 1.1 would require some very sophisticated new method in axiomatic set theory.

It is natural to ask now wether an affirmative answer to 1.1 could obtained for special classes of Hausdorff spaces. Our next two results are of this form. Let P denote the class of all hereditarily paracompact T_3 spaces.

1.6. THEOREM. If $X \in$ P and $|X| \geq \omega$, then $o(X)=o(X)^{\omega}$.

Proof. Suppose, on the contrary, that $\varkappa=o(X) < \varkappa^{\omega}$. Similarly as in the proof of 1.5 we let λ be the smallest cardinal whose ω^{th} power exceeds $\varkappa$. Then $cf(\lambda)=\omega < \lambda \leq \varkappa$. We can of course assume that for all $Y \subset X$ with $o(Y)=\varkappa$ we have $\sigma(Y)=\sigma(X)$. Since P$\subset\mathcal{K}$, and the class P is hereditary, the same argument as in the proof of 1.5 yields then that $\sigma=\sigma(X) < \lambda$. Put $\rho=\min\{\alpha:\sigma^{\alpha} > \lambda\}$. By the choice of λ then $\rho > \omega$. The following claim is the crux of the proof.

Claim. Let $\langle \varkappa_{\xi} : \xi < \rho \rangle$ be a sequence of cardinals such that $\varkappa_{\xi} < \sigma$ for every $\xi < \rho$. Then there is a disjoint family $\{G_{\xi}:\xi < \rho\}$ of sets open in X such that $\varkappa_{\xi} < o(G_{\xi})$ for each $\xi < \rho$. In particular $\Pi\{\varkappa_{\xi}: \xi < \rho\} \leq o(X)=\varkappa$.

Proof of the claim. Clearly we have a locally finite open cover U of X for which $o(\bar{U}) \leq \sigma$ for every $U \in$ U. Now we define by transfinite induction for $\xi < \rho$ open sets $G_{\xi} \subset X$ and $U_{\xi} \in$ U such that $G_{\xi} \subset U_{\xi}$. Suppose that $\eta < \rho$ and G_{ξ}, U_{ξ} have been defined for $\xi < \eta$. Then

$$o(\cup\{\bar{U}_{\xi}: \xi < \eta\}) \leq \sigma^{|\eta|} < \lambda \leq \varkappa,$$

hence for $Y=X \setminus \cup\{\bar{U}_{\xi}: \xi < \eta\}$ we have $o(Y)=\varkappa$. Since U is locally finite Y is open, moreover $\sigma(Y)=\sigma$ by our assumption. Thus there is $p \in Y$ for which $\sigma(p,Y)=\sigma(p,X) > \varkappa_{\eta}$. Now pick $U_{\eta} \in$ U such that $p \in U_{\eta}$, and put $G_{\eta}=Y \cap U_{\eta}$. Then $p \in G_{\eta}$ implies $o(G_{\eta}) \geq \sigma(p,X) > \varkappa_{\eta}$, and clearly $\xi < \eta$ implies $G_{\xi} \cap G_{\eta}=\emptyset$. The claim is thus proven.

An immediate consequence of this claim is that $\tau < \sigma$ implies $\tau^{\rho} \leq \varkappa$, and thus $\tau^{\rho} < \lambda$ as well (indeed, $\tau^{\rho} \geq \lambda$ would imply $\tau^{\rho} \geq \lambda^{\omega} > \varkappa$). Consequently the power σ^{ρ} is a jump, hence $\rho=cf(\sigma)$ by 1.3. Now write $\sigma=\sum\{\varkappa_{\xi}: \xi < \rho\}$, where $\varkappa_{\xi} < \sigma$ for each $\xi < \rho$. Applying the claim to the sequence $\langle \varkappa_{\xi}: \xi < \rho \rangle$ we get a disjoint open family $\{G_{\xi}: \xi < \rho\}$ such that $o(G_{\xi}) > \varkappa_{\xi}$ for $\xi < \rho$. But then by 1.2 *c)*

$$\sigma^{\rho}=\Pi\{\varkappa_{\xi}: \xi < \rho\} \leq \Pi\{o(G_{\xi}) : \xi < \rho\} \leq o(X) = \varkappa < \lambda^{\omega},$$

while clearly $\sigma^{\rho} > \lambda$ implies $\sigma^{\rho} \geq \lambda^{\omega}$, a contradiction, which completes our proof.

Now let G be the class of all T_2 topological groups.

1.7. THEOREM. Let $G \in$ G, $|G| \geq \omega$. Then $o(G)=o(G)^{\omega}$.

Proof. Let e denote the unit element of G, V be the neighbourhood filter of e in G, and put $\sigma=\sigma(e,G)=\sigma(G)$. We have to distinguish two cases:

Case a. There is $V \in$ V such that $o(V)=\sigma$ and finitely many left tranlates of V cover G, i.e. there is a finite set $A \subset G$ for which $G=\cup\{aV: a\in A\}$. Clearly then $o(G) \leq \Pi\{o(a\cdot V): a\in A\}=\sigma$, while G contains an infinite disjoint family $\{H_n: n\in\omega\}$ of non-empty open sets, hence by $o(H_n) \geq \sigma$ we have $o(G) \geq \sigma^{\omega}$ and consequently $o(G)=o(G)^{\omega}$.

Case b. There is no $V \in$ V as in case a. Let $U \in$ V be arbitrary with $o(U)=\sigma$ and pick a symmetric neighbourhood $V \in$ V such that $V^2 \subset U$. Consider $A \subset G$ such that $\{aV: a\in A\}$ forms a maximal disjoint family of left translates of V. We claim that $\cup\{aU: a\in A\}= G$. Indeed for any $x\in G$ there is $a\in A$ with $(xV)\cap(aV)\neq\emptyset$, hence there are $v_1, v_2 \in V$ such that $xv_1=av_2$. Then $x=av_2v_1^{-1}$, and $v_2v_1^{-1}\in U$ implies $x\in aU$.

Thus by our assumption $|A|=\alpha \geq \omega$, and obviously

$$o(G) \leq \Pi\{o(aU): a\in A\}= \sigma^{\alpha}$$

on the one hand and

$$o(G) \geq \Pi\{o(aV): a\in A\} = \sigma^{\alpha}$$

on the other. But then $o(G)=\sigma^{\alpha}=(\sigma^{\alpha})^{\omega}$.

I would like to mention at the end of this section the following problem.

1.8. PROBLEM. If X is an infinite compact T_2 space, is $o(X)= o(X)^{\omega}$?

§.2. OMITTING CARDINALS BY COMPACT SPACES

The problems considered in this section are motivated by [10], where the following question is investigated *(under GCH):* does every Lindelöf space of cardinality ω_2 contain a Lindelöf subspace of cardinality ω_1? There it is also shown that any uncountable compact T_2 space contains a Lindelöf subspace of cardinality ω_1. This leads naturally to the following definition.

2.1. DEFINITION. The compact T_2 space X is said to omit the infinite cardinal $\varkappa$ if $|X| > \varkappa$ and X contains no closed *(=compact)* subspace of cardinality $\varkappa$.

EXAMPLE 1. It is well-known that βN omits every infinite $\varkappa < 2^{2^{\omega}}$.

The following example is due to E.van Dowen and it is included here with his kind permission.

EXAMPLE 2. Let λ be a strong limit cardinal with $cf(\lambda)=\omega$, and X be a compact T_2 space such that $|X| > \lambda$ and every countable discrete subset of X is C^*-embedded (e.g. X is an F-space, cf [1]). Then X omits λ.

It is enough to show that $Y\subset X$ and $|Y|=\lambda$ implies $|\bar{Y}| \geq \lambda^{\omega}(>\lambda)$. In fact we can restrict ourselves to discrete subspaces, because by 3.2 of [13] any such Y contains a discrete subspace of size λ. Now let A be a family of almost disjoint ω-element subsets of Y with $|\mathsf{A}|=\lambda^{\omega}$ (cf. [1] or [16]). It is easy to see that if $A,B\in\mathsf{A}$ and $A\neq B$ then no limit point of A is a limit point of B, hence clearly $|\bar{Y}| \geq |\mathsf{A}|=\lambda^{\omega}$.

The above two examples tell all what is known about cardinals that *can be* omitted by a compact T_2 space.

In what follows we always assume GCH. Next we want to formulate a result showing that the omitting of cardinals by compact spaces is subject to some strict limitations. First however we prove a lemma which is interesting in itself.

2.2. LEMMA. *(GCH)* Suppose X is compact T_2 and omits $\varkappa^+$; then there is a closed subspace $F\subset X$ with a point $p\in F$ such that $\chi(p,F)=\varkappa^+$.

Proof. We can of course assume that X has a dense subset of size $\varkappa^+$ and that $\chi(p,X)\neq\varkappa^+$ for each $p\in X$. Then by 2.20 of [13] we have $|\{p\in X:\ \chi(p,X) \leq \varkappa\}| \leq (\varkappa^+)^{\varkappa}=\varkappa^+$, hence $\{p\in X :\ \chi(p,X) \geq \varkappa^{++}\}$ is a $G_{\varkappa^+}$-set in X *(i.e. the intersection of* $\varkappa^+$ *open sets)*, and therefore it contains a closed non-empty $G_{\varkappa^+}$-set Z. It is easy to see that

$\chi(p,Z) \geq \varkappa^{++}$ is valid for all $p\in Z$, hence we can apply proposition 2 of [10] to find a set $A\subset Z$ with $|A|=2^{\varkappa}=\varkappa$ such that $|\bar{A}| > \varkappa$. Since X omits $\varkappa^+$, we actually have $|\bar{A}| > \varkappa^+$ as well. Now let $F=\bar{A}$. Then $w(F) \leq 2^{|A|}=\varkappa^+$, hence $\chi(p,F) \leq \varkappa^+$ for every $p\in F$. On the other hand by 2.20 of [13] again $|\{p\in F: \chi(p,F) \leq \varkappa\}| \leq \varkappa^{\varkappa}=\varkappa^+ < |F|$, hence we must have a point $p\in F$ with $\chi(p,F)=\varkappa^+$.

2.3. THEOREM. (GCH) A compact T_2 space X cannot omit both $\varkappa^+$ and $\varkappa^{++}$.

Proof. Suppose that $|X| > \varkappa^{++}$ and X omits $\varkappa^+$. Then by lemma 2.2 there is a closed subsapce $F\subset X$ with a point $p\in F$ such that $\chi(p,F)=\varkappa^+$. One can easily construct then a strictly decreasing sequence $\{K_\nu: \nu < \varkappa^+\}$ of closed subsets of F such that $\cap\{K_\nu: \nu<\varkappa^+\}=\{p\}$. For each $\nu < \varkappa^+$ pick a point $p_\nu\in K_\nu\setminus K_{\nu+1}$. Clearly then

$$\overline{\{p_\nu: \nu < \varkappa^+\}} = \bigcup_{\alpha<\varkappa^+} \overline{\{p_\nu: \nu < \alpha\}} \cup \{p\}.$$

If there is an $\alpha < \varkappa^+$ such that $\overline{\{p_\nu: \nu < \alpha\}}$ has cardinality $\varkappa^{++}$, we are done. If not, i.e. if $|\overline{\{p_\nu: \nu < \alpha\}}| \leq \varkappa^+$ for each $\alpha < \varkappa^+$, then clearly $|\overline{\{p_\nu: \nu < \varkappa^+\}}|=\varkappa^+$, which is impossible since X omits $\varkappa^+$.

Finally we mention the following simple result.

2.4. THEOREM. (GCH) A compact T_2 space cannot omit both ω and ω_2.

Proof. This follows immediately from theorem 2 of [10] which says that if a compact T_2 space X omits ω then $|X| \geq 2^{\omega_1}$.

The following two simplest questions remain open:

2.5. PROBLEM (GCH) Can a compact T_2 space omit ω_2, or ω_1 and ω_3?

REFERENCES

[1] W.W. Comfort and S. Negrepontis, The theory of ultrafilters, Springer-Verlag, New York-Heidelberg-Berlin, 1974.

[2] B. Efimov, On extremally disconnected bicompacta, Dokl. Akad. Nauk SSR, 172(1967), 771-774. (in Russian)

[3] J. de Groot, Discrete subspaces of Hausdorff spaces, Bull. Acad. Pol. Sci.(13 1965), 537-544.

[4] A. Hajnal, Proof of a conjecture of S. Ruziewicz, Fund. Math. 50(1961/62), 123-128.

[5] A. Hajnal and I. Juhász, Discrete subspaces of topological spaces, II, Indog. Math. 31(1969), 18-30.

[6] A.Hajnal and I. Juhász, A consistency result concerning hereditarily α-Lindelöf spaces, Acta Math. Acad. Sci. Hung.

[7] A.Hajnal and I. Juhász, A consistency result concerning hereditarily α-separable spaces, Indag. Math. 35(1973), 301-307

[8] A. Hajnal and I. Juhász, A separable normal topological group need not be Lindelöf, Gen. Top. Appl. 6(1976), 199-205

[9] A. Hajnal and I. Juhász, On the number of open sets, Ann. Univ. Sci. Budapest, 16(1973), 99-102

[10] A. Hajnal and I. Juhász, Remarks on the cardinality of compact spaces and their Lindelöf subspaces, Proc. AMS (to appear)

[11] T. Jech, Lectures in set theory, Springer Lecture Notes in Math., Vol. 217, Berlin-Heidelberg-New York, 1971

[12] R.B. Jensen and K. Devlin, Marginalia to a Theorem of Silver. (to appear)

[13] I. Juhász, Cardinal functions in topology, Math. Centre Tract. No. 34, Amsterdam, 1971

[14] M. Magidor, On the singular cardinals problem, I. (to appear)

[15] R.S. Pierce, On a set-theoretic problem, Proc. Amer. Math. Soc., 9(1958), 892-896

[16] A. Tarski, Sur la décomposition des ensembles en sous-ensembles presque dijoints, Fund. Math. 12(1928), 188-205.