Jürgen Flachsmeyer
Topologization of Boolean algebras

In: (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part A: Invited papers. Springer, Berlin, 1977. Lecture Notes in Mathematics, 609. pp. 81--97.

Persistent URL: http://dml.cz/dmlcz/700998

TOPOLOGIZATION OF BOOLEAN ALGEBRAS

J. Flachsmeyer

Sektion Mathematik Ernst-Moritz-Arndt Universität
Jahnstrasse 15a, 22 Greifswald, DDR

Introduction

We start with the standard facts in the category of Boolean rings, Boolean lattices and Boolean spaces. Then we define the hyperstonian cover induced by the second function dual of a compact space and show the connections with the Gleason cover. Section 2 contains the notion of a Boolean ring of idempotents of a ring and some relevant examples. Theorem 4 identifies the Stone representation spaces for the Boolean algebra of all bands in the case of the function lattice C(S) and the measure lattice M(X). (Part I of this theorem seems to be foklore (?); but part II seems to be new, for the special case of Boolean spaces being a consequence of the investigations of D. A. EDWARDS). In Section 3 we show how one can obtain compatible Boolean ring topologies by algebra lattice topologies on the function space of the corresponding representation space. An important procedure which yields algebra lattice topologies arises by working with measures. (Theorem 6 on Riesz topologies.) For Boolean algebras the corresponding topologies are the Nikodym topologies (Corollary of Theorem 6). Such topologies on Boolean algebras were studied by J. L. B. COOPER and D. A. EDWARDS. The concluding remarks concern topological semifields in the sense of ANTONOVSKI-BOLTJANSKII-SARYMSAKOV. Our main point of view - the interplay of Boolean ring topologies and function topologies - has here its natural action.

1. Preliminaries

1.1. The Stone functor

For the basic facts concerning Boolean algebra we refer to SIKORSKI [24], HALMOS [16] and SEMADENI [23]. It is possible to develop Boolean theory from several points of view, one of which is of algebraic character (Boolean rings), another one of order-theoretic type (Boolean lattices), while the third one is of topological nature (Boolean spaces). The mutual interplay of these three directions was strongly emphasized by the fundamental work of M. H. STONE [25]. With every Boolean ring a Boolean lattice is associated and vice versa. This correspondence effects an isomorphy between the category of all Boolean rings BoolRg, i.e. the idempotent rings with their ring homomorphisms, and the category of all Boolean lattices BooLat, i.e. the distributive relatively complemented lattices with their lattice homomorphisms preserving the relative complement. In the case of Boolean rings with unit the restriction to such homomorphisms is necessary which preserve the unit. This yields the subcategory BoolRgI. An equivalent lattice category is that of BoolAlg, where the Boolean algebras, i.e. distributive complemented lattices, are the objects and the lattice homomorphisms preserving the complement are the morphisms. The prototype of the natural correspondence of Boolean lattices and Boolean rings is that of a field of sets (=algebra of sets) with the operations of union, intersection and set-complementation and the same family of sets with the operations of symmetric difference and intersection. Every Boolean algebra (resp. lattice) can be regarded as a field of sets (resp. ring of sets). By the important Stone representation theorem to every Boolean algebra (Boolean lattice) $\mathbb{B}$ there corresponds a topologically unique Hausdorff zero-dimensional compact (locally compact) space- the so called spectrum Spec($\mathbb{B}$) of $\mathbb{B}$ - such that $\mathbb{B}$ is isomorphic to the field of all clopen subsets of Spec($\mathbb{B}$) (resp. the ring of all compact-open subsets). The field of all clopen subsets of a Boolean space X (=Hausdorf zero-dimensional compact space) is called the dual algebra $\mathbb{D}$ (X) of X . The Stone representation produces a dual equivalence of the category BoolAlg and the category $\underline{\mathrm{Comp}_0}$ of Boolean spaces and their continuous maps.

$$\underline{\mathrm{BoolAlg}} \underset{\text{Dual algebra}}{\overset{\text{Stone functor}}{\rightleftarrows}} \underline{\mathrm{Comp}_0}$$

For the case of BoolRg resp. BoolLat it is convenient to exclude the zero-ring resp. the zero-lattice and to make restriction to proper homomorphisms (=non-zero homomorphisms) only. Then the Stone representation yields a dueal equivalence of BoolRg (resp. BoolLat) to the category $\underline{\mathrm{LocComp}_0}$ of locally compact zero-dimensional spaces and proper maps (=continuous closed maps with compact fibres).

1.2. The hyperstonian cover induced by the second function dual of a compact space

If the Stone functor is used, two Banach lattices can be immediately associated with each Boolean algebra $\mathbb{B}$, the space $C(\mathrm{Spec}(\mathbb{B}))$ of all continuous realvalued functions on $\mathrm{Spec}(\mathbb{B})$ and the space $M(\mathrm{Spec}(\mathbb{B}))$ of all bounded regular Borel measures on $\mathrm{Spec}(\mathbb{B})$. By the Riesz integral representation theorem it is wellknown that for every $X \in$ Comp(=category of all compact Hausdorff spaces and their continuous maps), $M(X)$ is the Banach dual of $C(X)$. $M(\mathrm{Spec}(\mathbb{B}))$ is the same as the Banach lattice $M(\mathbb{B})$ of all bounded finitely additive measures on $\mathbb{B}$ (cf. EDWARDS [9]). While $M(X)$ is always a complete vector lattice (because it is the order dual of $C(X)$) , $C(X)$ becomes a complete vector lattice iff X is a Stonian space, i.e. every open subset has open closure (= an extremally disconnected space) (Stone-Nakano theorem [26], [19]). It is possible to assign each $X \in$ Comp two Stonian spaces. The first one is the spectrum of the complete Boolean algebra $R_o(X)$ of all regular open sets of X , moreover, there is a canonical continuous map of $\mathrm{Spec}(R_o(X))$ onto X . This situation is characterized as follows (cf. GLEASON [15]):
For every $X \in$ Comp there is a unique (up to a topological isomorphism) Stonian space gX with a continuous irreducible map onto X (i.e. no proper closed subset is mapped onto X)

$$g: gX \longrightarrow X .$$

This mapping situation (resp. the preimage space gX) will be called the Gleason cover of X . (It is also usual to call it the Gleason resolution or the projective resolution of X). From a more functional analytic point of view the Gleason cover arises as the structure space of the Dedekind-MacNeille completion of the vector lattice $C(X)$ (cf. DILWORTH [6], and for another approach FLACHSMEYER [13]).

The second Stonian space associated with X can be obtained as follows:
The second Banach dual $C''(X)$ of $C(X)$ has the Kakutani representation $C(X)$ as a complete M-space over a Stonian space $\tilde{X}$. The canonical injection of $C(X)$ in its second dual then induces a continuous surjection from $\tilde{X}$ onto X . This mapping situation (resp. the preimage space) will be called the hyperstonian cover of X

$$h: hX \longrightarrow X$$

induced by the second function dual. The space hX is hyperstonian in the sense of DIXMIER [7]. This means that the union of the supports of all hyperdiffuse measures $\nu \in M(hX)$ forms a dense subset of hX. A positive measure $\nu \in M^+(hX)$ is said to be <u>hyperdiffuse</u> (or normal) iff every nowhere dense Borel set has ν-measure zero. In general ν is hyperdiffuse iff ν^+ and ν^- are hyperdiffuse. The canonical embedding of $M(X)$ into its second topological dual $M''(X) \cong M(hX)$ gives precisely the hyperdiffuse measures on hX.

<u>Problem</u>

Is there a nice topological descriptive characterization of the hyperstonian cover $h : hX \to X$ induced by the second function dual of X? This question is far from being answered.
Here we only want to compare the two covers $g : gX \to X$ and $h : hX \to X$. For this reason we give the following new characterization of the projective objects in <u>Comp</u>.

<u>Theorem 1</u>

A space $X \in$ <u>Comp</u> is Stonian iff every continuous surjection $f : Y \to X$ for $Y \in$ <u>Comp</u> is a retraction, i.e. there is an embedding $e : X \to Y$ such that $e \circ f$ is the identity on $e(X)$.

<u>Proof</u>

The extremal disconnectednes follows easily from the retraction property, because under this assumption the Gleason cover $g : gX \to X$ must be a retraction: $e \circ g|_{e(X)} = id_{e(X)}$. By the irreducibility of g the map e must be a topological isomorphism between gX and X. Conversely let X be extremally disconnected and let $f : Y \to X$ be a given epimorphism in <u>Comp</u>. With the help of Zorn's lemma f can be reduced to a closed subspace Y_o of Y, $f_o : Y_o \to X$, $f_o \equiv f|_{Y_o}$, such that f_o is irreducible. The projectivity of an extremally disconnected space implies that an irreducible surjection onto such a space has to be a topological isomorphism (see [15] for <u>Comp</u> or [10] for a much greater category). Now let e be the inverse map of f_o from X onto Y_o. Then $e \circ f$ is the identity on Y_o, thus f is a retraction. □

<u>Theorem 2</u>

Let $f : S \to X$ be a continuous surjection of a Stonian space S onto a given compact Hausdorff space X. Then every restriction $f_o : S_o \to X$ of f to an irreducible map f_o on the closed subspace $S_o \subset S$ is isomorphic to the Gleason cover.

Proof

Let $f_o : S_o \to X$ be a restriction of the above mentioned type. The projectivity of S yields the mapping diagram

$$\begin{tikzcd} S \arrow[rr, dashed, "\hat{f}"] \arrow[drr, "f"'] & & gX \arrow[d, "g"] \\ & & X \end{tikzcd}$$

Because of irreducibility of g the map f must be onto (hence, by the preceding theorem, f is a retraction). The irreducible restriction $f_o := f|_S$ induces an irreducible $\hat{f}_o := \hat{f}|_{S_o}$. Thus $\hat{f}_o$ is a topological isom8rphism. □

Corollary

For every $X \in$ Comp the hyperstonian cover $h : hX \to X$ induced by the second function dual contains the Gleason cover. Any irreducible restriction of h to a closed subspace of hX yields the Gleason cover.

Remark

The situation is clearly arranged for the case $\alpha \mathbb{N}$ - the Alexandrov-one-point compactification of the natural numbers.

The Gleason cover is established by the Stone-Čech-compactification with the natural projection map onto $\alpha \mathbb{N}$. The hyperstonian cover is given by the projection of the Stone-Čech-compactification $\beta \overline{\mathbb{N}}_{dis}$ of the discrete space $\mathbb{N} \cup \{\infty\}$ onto $\alpha \mathbb{N}$. In general, the hyperstonian cover $h : hX \to X$ contains even the part consisting of the natural projection of βX_{dis} onto X. (This part is associated with the discrete measures on X).

2. Some Boolean algebras of functional - analytic character

2.1. Boolean rings of idempotents of a ring

Let $\mathcal{R} = (R,+,\cdot)$ be a (not necessarily commutative) ring. The set $I(\mathcal{R}) = \{x : x^2 = x, x \in R\}$ of all idempotents of $\mathcal{R}$ has the following properties, which are easy to prove:

(1) $0 \in I(\mathcal{R})$.

(2) If $x, y \in I(\mathcal{R})$ and x, y commute, i.e. $xy = yx$, then $x \cdot y \in I(\mathcal{R})$.

(3) If $x, y \in I(\mathcal{R})$ and x, y commute, then $x + y - x \cdot y \in I(\mathcal{R})$, $x + y - 2xy \in I(\mathcal{R})$.

(4) If $\mathcal{R}$ has a unit element 1, then for $x \in I(\mathcal{R})$ also $1 - x \in I(\mathcal{R})$.

(5) A natural partial order can be introduced in $I(\mathcal{R})$ by the following definition: $x, y \in I(\mathcal{R})$, $x \leq y :\Leftrightarrow xy = yx = x$. With respect to this natural partial order, $I(\mathcal{R})$ has the following properties: Every pair x, y of commuting idempotents has an infimum
$\inf(x,y) = xy$
and supremum
$\sup(x,y) = x + y - xy$.

A <u>Boolean ring</u> $\mathbb{B}$ <u>of idempotents of</u> $\mathcal{R}$ is defined to be a nonvoid commuting set $\mathbb{B} \subset I(\mathcal{R})$ which is a Boolean ring with respect to the following operations

$$x \boxplus y = x + y - 2xy ,$$
$$x \boxdot y = xy .$$

The above quoted properties imply by virtue of Zorn's lemma:

(6) Every non-void commuting subset $K \subset I(\mathcal{R})$ is contained in a maximal one and every maximal commuting subset of idempotents forms a Boolean ring of idempotents of $\mathcal{R}$.
If $\mathcal{R}$ is a commutative ring, the set $I(\mathcal{R})$ of all idempotents is a Boolean ring of idempotents.
Special examples of this general procedure to get Boolean rings resp. algebras from (may be non-commutative) rings will be considered in the sequel in a more topological situation.

2.2. <u>Boolean rings of continuous indicator functions</u>

Let X be a topological space. Then the rings $C(X)$ and $C_b(X)$ of all continuous real-valued (resp. bounded c.r.v.) functions on X yield the same Boolean ring of all idempotents. Of course, a function $f \in C(X)$ is an idempotent iff f takes only values 0 or 1 and the support $\operatorname{supp} f$ is a clopen set in X .

The representation theorem of Stone can be formulated in the terminology of functions as follows:
For every abstract Boolean ring $\mathbb{B}$ there exists a unique (up to a homeomorphism) locally compact Boolean space X such that $\mathbb{B}$ is isomorphic to the Boolean ring of all idempotents of the ring $C_{oo}(X)$ of all continuous real-valued functions with compact support.

2.3. Boolean rings of projections in Banach spaces

Let E be a Banach space and $L(E, E)$ the algebra of all continuous linear operators on E (in general not commutative). The idempotents of the ring $L(E, E)$ are usually called projections. Thus, a Boolean ring of projections is to be understood as a Boolean ring of idempotents of the ring $L(E, E)$. This notion is equivalent to that used in the theory of self-adjoint and normal operators in Hilbert spaces and its generalization to Banach spaces. (cf. N. DUNFORD - J. SCHWARTZ [8], W. G. BADE [2]).

Every Boolean algebra can be realized as a Boolean algebra of projections in the Banach space $E = C(\mathrm{Spec}(\mathcal{B}))$.
The desired isomorphism is given by the correspondence

$$b \in \mathcal{B} \longmapsto P_b \in L(E, E) \text{ , where } P_b(f) = \chi_{U(b)} \cdot f$$

for $f \in C(\mathrm{Spec}(\mathcal{B}))$, $\chi_{U(b)}$ being the indicator function of the clopen set $U(b)$ which corresponds to b.

From the preceding observation we get by a dual argument that the Boolean algebra $\mathcal{B}$ can be represented as a Boolean algebra of projections in the Banach space $M(\mathrm{Spec}(\mathcal{B}))$ of all regular bounded Borel measures on the representation space of $\mathcal{B}$. The adjoint operator for the projection P_b in $C(\mathrm{Spec}(\mathcal{B}))$ is the projection

$$P_b^*(\mu) = \chi_{U(b)} \cdot \mu \qquad \text{for all} \quad \mu \in M(\mathrm{Spec}(\mathcal{B})) .$$

2.4. Boolean algebra of all bands of an order complete vector lattice

Let V be an order complete vector lattice. A band B in V is defined to be an order convex linear subspace (= l-ideal in V) with the property that B is supremum closed in V. Bands are sometimes called closed l-ideals or in Russian literature (e.g. [28]) components of V. Bands are the kernels of full order homomorphisms (i.e. sup(inf) stable).

V satisfies the so-called Riesz decomposition theorem: For $A \subset V$ the set $A^{\perp} = \{b : b \in V \ |b| \wedge |a| = 0 \text{ for all } a \in A\}$ is a band, $A^{\perp\perp}$ is the band generated by A and $V = A^{\perp\perp} \oplus A^{\perp}$. The system Band (V) of all bands of V ordered by inclusion forms a complete Boolean algebra, for which the lattice operations are the following:

$$(1) \quad B_1 \wedge B_2 = B_1 \cap B_2 ,$$

$$(2) \quad B_1 \vee B_2 = B_1 + B_2 \quad (=\mathrm{span}\ (B_1 \cup B_2)) ,$$

$$(3) \quad B' = B^{\perp} ,$$

$$(4) \bigwedge_{i\in \mathcal{J}} B_i = \bigcap_{i\in \mathcal{J}} B_i .$$

There is a 1-1 correspondence between the bands B of V and a special kind of projections $P : V \to V$, which are called full order projections and which are defined to be linear operators with the following properties:
(1) $P^2 = P$ and (2) $P(\sup A) = \sup P(A)$ for every bounded $A \subset V$.
This correspondence is established in the following way:
The projection P_B determined by the band B is the projection having B as range and annihilating its complement $B^\perp$.

$$B = \operatorname{Im} P_B , \quad B^\perp = \operatorname{Ker} P_B .$$

For every $x \in V$ $x \geqq 0$ it holds

$$P_B(x) = \sup \{ y : y \in B , \quad 0 \leqq y \leqq x \} \quad (*) .$$

Theorem 3

Let V be a complete vector lattice. The set $\mathcal{P}(V) \subset L(V, V)$ of all full order projections of V forms a complete Boolean algebra with respect to the operations

$$(1) \quad P \wedge Q = P \cdot Q = Q \cdot P ,$$
$$(2) \quad P' = I - P ,$$
$$(3) \quad P \vee Q = P + Q - P \cdot Q .$$

Further, it holds: The lattice order in $\mathcal{P}(V)$ is the same as the canonical projection order:

$$(4) \quad P \cdot Q = P \Leftrightarrow P \leqq Q \text{ (i.e. } Px \leqq Qx \text{ for all } x \geqq 0 \text{ in } V \text{)} \Longleftrightarrow \operatorname{Im} P \subset \operatorname{Im} Q .$$

The Boolean algebra $\mathcal{P}(V)$ is isomorphic to the Boolean algebra Band (V) given by the correspondence $P \mapsto \operatorname{Im} P$.

Proof

First we show that all full order projections commute and $\mathcal{P}(V)$ is closed under multiplication. $P \in \mathcal{P}(V) \mapsto \operatorname{Im} P \in \text{Band }(V)$ is a 1-1 map and onto. It suffices to prove $P_{B\cap C}(x) = P_B(P_C(x))$ for all $B, C \in \text{Band}(V)$ and all $x \geqq 0$, which follows straightforward from $(*)$.

Furthermore, $\mathcal{P}(V)$ is closed with respect to $P \mapsto I - P$ for $P \in \mathcal{P}(V)$ since $I - P_B = P_{B^\perp}$. Thus $\mathcal{P}(V)$ is a Boolean ring of idempotents of the algebra $L(V, V)$. The Boolean operations are those described in Section 2.1.

Now we prove (4): The relation $P \leqq Q \Leftrightarrow \operatorname{Im} P \subset \operatorname{Im} Q$ can be seen easily.

The other part holds because $P_{B\cap C} = P_B \cdot P_C$. The remaining part of the theorem is clear. $\square$

In the case when V is an order complete Banach lattice, the full order projections $P \in \mathcal{P}(V)$ are continuous, thus the Boolean algebra $\mathcal{P}(V)$ is a Boolean algebra of Banach space projections. The continuity of $P \in \mathcal{P}(V)$ follows from the relations $P|x| = |Px|$ and $P|x| \leqq |x|$. Then $\|Px\| \leqq \|x\|$ and $\|P\| \leqq 1$. Two types of Banach lattices are of special interest, namely the Kakutani M-spaces and the Kakutani L-spaces, because it is possible to identify the structure spaces of their Boolean algebras of bands.

Theorem 4

1. Let S be a Stone space. Then the Boolean algebra Band $(C(S))$ is isomorphic to the dual algebra of X. The isomorphism is given by the correspondence

$$\text{Band } (C(S)) \ni B \mapsto \text{supp } B \quad (= \text{cl}(\bigcup \text{supp } f : f \in B)) .$$

2. Let X be a compact Hausdorff space. Then the Boolean algebra Band $(M(X))$ is isomorphic to the dual algebra of the hyperstonian cover hX of X. The isomorphism is given by the correspondence

$$\text{Band } (M(X)) \ni B \mapsto \text{supp } hB \quad (= \text{cl}(\bigcup \text{supp } \mu^h, \mu \in B)) .$$

Proof

1. S being a Stone space is equivalent to $C(S)$ being a complete M-space. For every $f \in C(S)$, supp f $(= \text{cl}\{x : f(x) \neq 0\})$ is a clopen set. The equivalence of $|f| \wedge |g| = 0$ and supp $f \cap$ supp $g = \emptyset$ is easy to verify. Thus, for a band $B \in C(S)$ the complement $B^{\perp}$ consists of all $f \in C(S)$ with supp $f \subset S \setminus \text{sup } B$. $B = B^{\perp\perp}$ implies $B = \{g : g \in C(S) , \text{ supp } g \subset \text{supp } B\}$. Every clopen set $U \subset S$ determines also a band $B = \{f : \text{supp } f \subset U\}$. Therefore the ordered set Band $(C(S))$ is isomorphic to the ordered set $\mathbb{D}(S)$.

2. Let $N(hX)$ be the space of all hyperdiffuse measures on hX. In 1.2. we noticed that $M(X) \cong N(hX)$. For every $\mu \in M(X)$ let μ^h be the corresponding hyperdiffuse measure on hX. Every μ^h on hX gives a completely additive measure on the dual algebra $\mathbb{D}(hX)$ of all clopen sets of hX and vice versa (cf. [12]). Now for every σ-Boolean algebra $\mathbb{B}$ and for arbitrary positive σ-measures λ, ν on $\mathbb{B}$ it holds

$$\nu \in \{\lambda\}^{\perp\perp} \Leftrightarrow \nu \ll \lambda \quad (\text{i.e. } \lambda(b) = 0 \Rightarrow \nu(b) = 0, \ b \in \mathbb{B})$$

([23, p. 328]).

Thus $\mu, \nu \in M(X)$ implies: $|\mu| \wedge |\nu| = 0 \Leftrightarrow \text{supp } \mu^h \cap \text{supp } \nu^h = \emptyset$. $\square$

3. Compatible topologies on Boolean rings

3.1. Locally solid topologies

From topological algebra it is well known how to define the notion of a topological ring (see BOURBAKI [3]). Let $(R,+,\cdot)$ be a ring. A topology $\mathcal{O}$ on R is compatible with the algebraic structure if $(R,+,\mathcal{O})$ is a topological group and if the map $(a, b)\mapsto ab$ from $(R,\mathcal{O})\times(R,\mathcal{O})$ into $(R,\mathcal{O})$ is continuous. A ring endowed with a compatible topology is called a topological ring.

Using neighborhoods this can be described as follows: $(R,+,\cdot,\mathcal{O})$ is a topological ring iff the following conditions are satisfied: (N_1) For arbitrary points $x, y \in R$ and an arbitrary neighborhood W of $x - y$ there exist neighborhoods U of x and V of y with $U - V \subset W$. (N_2) For arbitrary points $x, y \in R$ and arbitrary neighborhood W of xy there exist neighborhoods U of x and V of y with $UV \subset W$. Due to the homogeneity of a topological ring the compatibility may be expressed by the neighborhood filter of the origin only. For a ring $(R, +, \cdot)$ to have a compatible topology $\mathcal{O}$ with a given filter base $\mathcal{B}$ on R as a neighborhood base of 0 with respect to $\mathcal{O}$ it is necessary and sufficient that the following conditions be satisfied:

NBZ 1. $0 \in B$ for all $B \in \mathcal{B}$.

NBZ 2. For every $B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ such that $C + C \subset B$.

NBZ 3. For every $B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ such that $-C \subset B$.

NBZ 4. For every $B \in \mathcal{B}$ and given $x \in R$ there is a $C \in \mathcal{B}$ with $xC \subset B$ and $Cx \subset B$.

NBZ 5. For every $B \in \mathcal{B}$ there is a $C \in \mathcal{B}$ such that $CC \subset B$.

Proposition

Let R be a Boolean ring and L the corresponding Boolean lattice. A topology $\mathcal{O}$ is compatible on R iff $\mathcal{O}$ is compatible with the lattice structure, i.e. $\mathcal{O}$ makes the maps $(x,y)\mapsto x\vee y$, $(x,y)\mapsto x\wedge y$ and $(x,y)\mapsto x - x\wedge y$ continuous.

Proof

This follows easily by observing that $x \vee y = x + y + xy$, $x\wedge y = xy$, $x - x\wedge y = x + xy$, $x + y = x\vee y - x\wedge y = (x - x\wedge y)\vee(y - x\wedge y)$. □

A topology $\mathcal{O}$ on a Boolean ring R is said to be locally solid (or locally order convex) iff every point has a neighborhood base of solid

(= order convex) sets

$$U:\ x,\ y \in U:\ x \leqq y \Rightarrow [x,\ y] = \{ z \,\vdots\, x \leqq z \leqq y \} \subset U\ .$$

Of course, a set $U \subset R$ containing 0 is solid iff $UR = U$ (the inclusion $U \subset UR$ for every non-void $U \subset R$ holds). It is easily obtained that a topology on R is locally solid at the origin iff the family of maps $\{ \varphi_a \,\vdots\, \varphi_a(x) := a \cdot x\ ,\ a \in R \}$ is equicontinuous at the origin.

Remark

Compatible topologies on Boolean rings with a solid neighborhood base of the origin are called uniformly compatible by J. L. COOPER [4],[5]. D. A. VLADIMIROV [27] used the notation monotone topology for such topologies while he called compatible topologies on Boolean algebras uniform topologies.

Theorem 5

Let X be a locally compact Boolean space and $C_{oo}(X)$ the lattice ring of all continuous real-valued functions with compact support on X. The locally solid compatible topologies on the Boolean ring I of idempotents of $C_{oo}(X)$ are exactly the trace topologies of all locally solid lattice algebra topologies on $C_{oo}(X)$.

Proof

Let $\mathcal{O}$ be a locally solid topology on $C_{oo}(X)$ with respect to which addition and multiplication in $C_{oo}(X)$ are continuous. Because the Boolean operations in I are related to the operations in $C_{oo}(X)$ by

$$f \boxplus g = f + g - 2fg\ ,\ f \boxdot g = f \cdot g$$

the trace topology on I is compatible. The I-trace of every solid set $U \subset C_{oo}(X)$ is solid in I .

Now let conversely a compatible topology $\mathcal{P}$ on I be given which is solid at the origin. We shall construct a locally solid lattice ring topology $\mathcal{O}$ on $C_{oo}(X)$ with the I-trace $\mathcal{P}$. Let U be a $\mathcal{P}$-neighborhood of 0 and $\varepsilon > 0$. We define $\widetilde{U}_\varepsilon := \{ f \,\vdots\, f \in C_{oo}(X)\ ,$ there is an $e \in U$ such that $(1-e) \cdot |f| < \varepsilon \}$.

Then the following conditions are satisfied:

(1) $\widetilde{U}_\varepsilon \cap \widetilde{V}_\varepsilon \supset \widetilde{(U \cap V)}_\varepsilon$.

(2) $\widetilde{U}_\varepsilon \subset \widetilde{U}_\delta$ for $\varepsilon < \delta$.

(3) $\alpha \widetilde{U}_\varepsilon \subset \widetilde{U}_\varepsilon$ for all reals α with $|\alpha| < 1$.

(4) For every $f \in C_{oo}(X)$ there exists an $\alpha \in \mathbb{R}$ such that $f \in \alpha \widetilde{U}_\varepsilon$.

(5) $\widetilde{U}_\varepsilon + \widetilde{U}_\varepsilon \subset \widetilde{(U \vee U)}_{2\varepsilon}$.

(6) $\widetilde{U}_\varepsilon \cdot \widetilde{U}_\varepsilon \subset \widetilde{(U \vee U)}_{\varepsilon^2}$.

(7) Every $\widetilde{U}_\varepsilon$ is absolutely order convex .

(8) For every $\varepsilon < 1$ $\widetilde{U}_\varepsilon \cap I = U \cdot I$.

Thus the set $\{ \widetilde{U}_\varepsilon : U$ $\mathcal{P}$-neighborhood of 0 in I , $\varepsilon > 0 \}$ forms a solid filter base of the origin in $C_{oo}(X)$ and this is the neighborhood filter base of a unique lattice algebra topology $\mathcal{O}$ on $C_{oo}(X)$. According to (8) the I-trace of $\mathcal{O}$ coincides with $\mathcal{P}$ for a topology $\mathcal{P}$ which is locally solid at the origin. □

Corollary

Every compatible Boolean ring topology which is locally solid at the origin is locally solid at each point.

Proof

This is no trivial statement because order and addition in a Boolean ring are not compatible. Nevertheless, the proof of the statement follows the lines of the preceding proof. The constructed topology for $C_{oo}(X)$ is locally solid at every point because order and addition are compatible in $C_{oo}(X)$. □

The following three extremal topologies for the algebra lattice $C_{oo}(X)$ offer themselves.

1. Norm topology 2. compact open topology and 3. simple topology.

By the norm topology in $C_{oo}(X)$ the discrete topology on the Boolean ring I of idempotents is induced. (The trace of every ε-ball in $C_{oo}(X)$ for $0 < \varepsilon < 1$ yields only the zero-element in I .)

Proposition

The compact open topology in $C_{oo}(X)$ generates a locally solid compatible Boolean ring topology on the Boolean ring I of idempotents. By interpretation of I as the ring of compact-open sets this topology on I has the closed-set topology $\mathcal{T}_K$ induced by all compact sets in the sense of the upper semifinite topology as an open neighborhood base at the origin. The uniform completion of the topological Boolean ring I yields the Boolean ring of all idempotents of the ring $C(X)$ of all continuous functions on X .

Proof

For topologies in the hyperspace of closed sets of a topological space X we refer to [18] and [11] . In [11] we considered the topology $\mathcal{T}_K$ for $F(X) = \{ F : F$ closed subset of $X \}$. $\mathcal{T}_K$ possesses a base given by the sets $\langle K\rangle = \{F : F \cap K = \emptyset \, , \, F \in F(X)\}$, K compact. The compact open topology on $C_{oo}(X)$ yields the following open neighborhood base for I at the origin:

$$U(K) = \{ u : u \in I \, , \, \operatorname{supp} u \cap K = \emptyset \}, \quad K \text{ compact.}$$

The Boolean ring $\tilde{I}$ of all idempotents of $C(X)$ is complete with respect to the compact open topology and I is dense in $\tilde{I}$. □

Proposition

The simple topology in $C_{oo}(X)$ induces a locally solid compatible Boolean ring topology on the Boolean ring I of idempotents. This topology is a precompact ring topology on I . The uniform completion of this topological Boolean ring I yields the so-called perfect completion of I .

Proof

Following F. B. WRIGHT [29] , the embedding of the Boolean ring I into the Boolean ring $\tilde{I}$ of all idempotents of $\mathbb{R}^X$ will be called the perfect completion. $\tilde{I}$ is compact with respect to the simple topology as the Tichonov product theorem shows. I is dense in $\tilde{I}$, since every indicator function of a single point is the pointwise limit of all elements of I whose support contains the point. Then every indicator function of an arbitrary non-void set S of X is the pointwise limit of all indicator functions with finite support contained in S . Thus the simple topology on I is a precompact ring topology which yields the perfect completion. □

3.2. The Riesz topology for functions and the induced Boolean ring topologies

What can be said about the possibility to get other algebra lattice topologies in $C_{oo}(X)$ for arbitrary locally compact spaces X ? Via the order dual of the Riesz space $C_{oo}(X)$ we have the possibility to introduce algebra lattice topologies. The order dual of $C_{oo}(X)$ is the vector lattice M(X) of all Radon measures on X . Let $\mu \in M(X)$. We consider the topology of convergence in measure for $C_{oo}(X)$. This topology in $C_{oo}(X)$ will be called the Riesz topology induced by μ , because the notion of convergence in measure goes back to F. RIESZ [22]. If we start with a family $F \subset M(X)$ the supremum of all Riesz topologies induced by $\mu \in F$ will be called the Riesz topology induced by F . Let us recall some facts about the Riesz topology. For every $\varepsilon > 0$ let

$$U_\varepsilon^\mu := \{f : f \in C_{oo}(X) \text{ with } |\mu|\left(\frac{|f|}{1+|f|}\right) < \varepsilon\}.$$

Then the following conditions are fulfilled.

(1) $U_\varepsilon^\mu \subset U_\delta^\mu$ for $0 < \varepsilon < \delta$,

(2) $\alpha U_\varepsilon^\mu \subset U_\varepsilon^\mu$ for all $|\alpha| < 1$,

(3) for every $f \in C_{oo}(X)$ there exists an $\alpha \in \mathbb{R}$ such that $\alpha f \in U_\varepsilon^\mu$,

(4) $U_\varepsilon^\mu + U_\varepsilon^\mu \subset U_{2\varepsilon}^\mu$,

(5) $U_\varepsilon^\mu \cdot U_\varepsilon^\mu \subset U_{2\varepsilon}^\mu$.

(6) Every U_ε^μ is absolute order convex.

Thus the Riesz topology on $C_{oo}(X)$ induced by the Radon measure μ is an algebra lattice topology which is pseudo-metrizable. The trace topology on the Boolean ring I of idempotents of $C_{oo}(X)$ is precisely the Nikodym topology on I induced by μ. For a Boolean ring R which admits a positive measure $\mu > 0$, NIKODYM [20],[21] introduced a pseudo-metric $\rho(a, b) = \mu((a \vee b) - (a \wedge b))$. If we start with a $\mu \in M(X)$, we get for idempotents φ, ψ of $C_{oo}(X)$

$$|\mu|\left(\frac{|\varphi - \psi|}{1+|\varphi-\psi|}\right) = \frac{1}{2}|\mu|(|\varphi-\psi|) = \frac{1}{2}|\mu|((\varphi \vee \psi) - (\varphi \wedge \psi)).$$

Theorem 6

Let X be a compact Hausdorff space. Two families $F_1, F_2 \subset M(X)$ of Radon measures on X generate the same Riesz topologies on C(X) iff the generated bands $B(F_1)$, $B(F_2)$ in M(X) coincide. Thus the Riesz topologies on C(X) with respect to the inclusion form a Boolean algebra which is isomorphic to the algebra of bands of M(X).

Proof

Let $F \subset M(X)$ and let $\hat{F}$ be the l-ideal generated by F:

$$\hat{F} = \{\nu : \nu \in M(X) \quad |\nu| \leq \sum_{i=1}^{n} \alpha_i \cdot |\mu_i| \quad \alpha_i \in \mathbb{R}^+, \quad \mu_i \in F\}.$$

(1) We will show for the R-topologies: $\mathcal{O}(F) = \mathcal{O}(\hat{F})$.

(2) Then we will show $\mathcal{O}(\mathrm{cl}(\hat{F})) = \mathcal{O}(\hat{F})$.
Because in every L-space the norm-closed l-ideals and the bands are the same objects we have by (1) and (2)
$$\mathcal{O}(F) = \mathcal{O}(B(F)).$$

(3) Then we prove for bands $B_1, B_2 \in \mathrm{Band}(M(X))$:
$$B_1 \subsetneq B_2 \Rightarrow \mathcal{O}(B_1) \subsetneq \mathcal{O}(B_2).$$

Ad (1): a) $U_\varepsilon^\mu \cap U_\varepsilon^\nu \subset U_{2\varepsilon}^{|\mu|+|\nu|}$,

b) $U_{\varepsilon}^{\alpha\mu} = U_{\frac{\varepsilon}{|\alpha|}}^{\mu} \quad \alpha \neq 0$,

c) $|\nu| \leqq |\mu| \Longleftrightarrow U_1^{\mu} \subset U_1^{\nu}$.

In c) the implication from the left to the right is obvious. Let $h \in C(X)$ with $0 \leqq h$ and $|\mu|(h) > 0$. Then for every $\alpha \in \mathbb{R}$ for which $0<\alpha<1$,

$$|\mu|\left(\frac{\alpha h}{|\mu|(h)}\right) < 1$$

thus $f \in U_1^{\mu}$ for $f \in C(X)$ with $\frac{\alpha h}{|\mu|(h)} = \frac{f}{1+f}$. By the assumption $f \in U_1^{\nu}$, i.e. $|\nu|\left(\frac{\alpha h}{|\mu|(h)}\right) < 1$. Therefore $|\nu|(h) \leqq |\mu|(h)$.

Ad (2): Let $\mu_n \in \hat{F}$ and $\|\mu_n - \mu\| \to 0$. Then $U_{\varepsilon}^{\mu_n} \subset U_{2\varepsilon}^{\mu}$ for $\|\mu_n - \mu\| < \varepsilon$, thus the topology $\mathcal{O}(\hat{F})$ is finer than the topology $\mathcal{O}(\mu)$.

Ad (3): It is clear that $\mathcal{O}(B_1) \subset \mathcal{O}(B_2)$ for $B_1 \subset B_2$. For $0 \leqq \nu \in B_2$ with $\nu \notin B_1$ we must prove $\mathcal{O}(\nu) \not\subset \mathcal{O}(B_1)$. In the case of $\mathcal{O}(\nu) \subset \mathcal{O}(B_1)$ we should have a measure $0 \leqq \mu \in B_1$ with $U_1^{\mu} \subset U_1^{\nu}$. Then by c) of (1) $0 \leqq \nu \leqq \mu$, which contradicts $\nu \notin B_1$. The proof is now complete since the bands of $M(X)$ form a Boolean algebra.□

Together with the theorem of Section 3.1 we get the following

<u>Corollary</u>

Let $\mathbb{B}$ be a Boolean algebra. Then the Nikodym topologies on $\mathbb{B}$ generated by families $F \subset M(\mathbb{B})$ of measures form with respect to inclusion a Boolean algebra which is isomorphic to the Boolean algebra of all bands of $M(\mathbb{B})$. The Nikodym topology on $\mathbb{B}$ generated by $F \subset M(\mathbb{B})$ is the trace of the Riesz topology in $C(\mathrm{Spec}(\mathbb{B}))$ induced by F .

3.3. <u>The underlying Boolean algebras of topological semifields</u>

In the theory of topological semifields founded by M. YA. ANTONOVSKI, V. G. BOLTJANSKII and T. A. SARYMSAKOV [1] complete Boolean algebras with compatible locally solid topologies are of special significance. Let us understand by an <u>ABS-Boolean algebra</u> such complete Boolean algebra $\mathbb{B}$ which is equipped with a Hausdorff compatible locally solid topology which is smaller than the order topology on $\mathbb{B}$. These and only these Boolean algebras are underlying Boolean algebras of topological semifields. If we start with a hyperstonian space X we get by the Nikodym topology $\mathcal{O}(N(X))$ (of all hyperdiffuse measures on X) on the dual algebra $\mathbb{D}(X)$ an ABS-topology. $C(X)$ with the Riesz topology $\mathcal{O}(N(X))$ will be a canonical associated topological semifield. For these results see a forthcomming paper of the author [14].

References

[1] M. YA. Antonovski, V. G. Boltjanskii and T. A. Sarymsakov: A survey on the theory of topologcal semifields. Uspehi Mat. Nauk 21 (1966), 185 - 218 . (Russian.)

[2] W. G. Bade: On Boolean algebras of projections and algebras of operators. Trans. Amer. Math. Soc. 80 (1955), 345 - 360.

[3] N. Bourbaki: Topologie générale. Paris 1951-58.

[4] J. L. Cooper: Convergence of families of completely additive set functions. Quart. J. Math. Oxford Ser. 20 (1949), 8 - 21.

[5] J. L. Cooper: Topologies in rings of sets. Proc. London Math. Soc. (2) 52 (1951), 220 - 239.

[6] R. P. Dilworth: The normal completion of the lattice of continuous functions. Trans. Amer. Math. Soc. 68 (1950), 427 - 438.

[7] J. Dixmier: Sur certains espaces considérés par M. H. Stone. Summa Brasil. Math. 2 (1951), 151 - 182.

[8] N. Dunford and J. T. Schwartz: Linear operators. Interscience Publ. 1958.

[9] D. A. Edwards: A class of topological Boolean algebras. Proc. London Math. Soc. (3) 13 (1963), 413 - 429.

[10] J. Flachsmeyer: Topologische Projektivräume. Math. Nachr. 26 (1963), 57 - 66.

[11] J. Flachsmeyer: Verschiedene Topologisierungen im Raum der abgeschlossenen Mengen. Math. Nachr. 26 (1963-64), 231 - 237.

[12] J. Flachsmeyer: Normal and category measures on topological spaces. Proc. Third Prague Topological Sympos. 1971, 109 - 116.

[13] J. Flachsmeyer: Dedekind-MacNeille extensions of Boolean algebras and of vector lattices of continuous functions and their structure spaces. General Topology and Appl. (To appear.)

[14] J. Flachsmeyer: Three examples of topological semifields. (To appear; Russian.)

[15] A. Gleason: Projective topological spaces. Illinois J. Math. 2 (1958), 482 - 489.

[16] P. Halmos: Lectures on Boolean algebras. Van Nostrand Math. Studies 1967.

[17] S. Kaplan: On the second dual of the space of continuous functions. Trans. Amer. Math. Soc. 86 (1957), 70 - 90.

[18] E. Michael: Topologies on spaces of subsets. Trans. Amer. Math. Soc. 71 (1951), 152 - 182.

[19] H. Nakano: Über das System aller stetigen Funktionen auf einem topologischen Raum. Proc. Imp. Acad. Tokyo 17 (1941), 308 - 310.

[20] O. Nikodym: Sur une generalisation des integrals de M. J. Radon. Fund. Math. 15 (1930), 131 - 179.

[21] O. Nikodym: Sur les suites convergentes de fonctions parfaitement additives d'ensemble abstrait. Monatsh. Math. 40 (1933), 427 - 432.

[22] F. Riesz: Sur les suites de fonctions mesurables. C. R. Acad. Sci. Paris Sér. A - B 148 (1909), 1303 - 1305.

[23] Z. Semadeni: Banach spaces of continuous functions. PWN Warszawa 1971.

[24] R. Sikorski: Boolean algebras. Springer 1960.

[25] M. H. Stone: Applications of the theory of Boolean rings to general topology. Trans. Amer. Math. Soc. 41 (1937), 375 - 381.

[26] M. H. Stone: Boundedness properties in function lattices. Canad. J. Math. 1 (1949), 176 - 186.

[27] D. A. Vladimirov: Boolean algebras. Izdat. Nauka, Moscow 1969. (Russian.)

[28] B. Z. Vulich: Introduction to the theory of semi-ordered spaces. Goz. Izdat., Moscow 1961. (Russian.)

[29] F. B. Wright: Some remarks on Boolean duality. Portugal Math. 16 (1957), 109 - 117.