ON DIMENSIONAL PROPERTIES OF INFINITE-DIMENSIONAL SPACES

Yu. SMIRNOV

Moscow

This report contains some results of myself and my pupils B. LEVSHENKO and E. SKLYARENKO, concerning infinite-dimensional spaces.

W. HUREWICZ was the first to obtain results in this area for separable metric spaces.

H. Theorem 1. *If a space R has small transfinite dimension* ind R *then* ind $R < \omega_1$.

H. Theorem 2. *The Hilbert cube J^∞ has no transfinite dimension* ind.

J. NAGATA calls a space R countable-dimensional when R is a countable union of zero-dimensional sets N_i, i. e. $R = \cup N_i$, $\dim N_i = 0$.

H. Theorem 3. *Let R be a space with a complete metric; then R has small transfinite dimension* ind R *if and only if R is countable-dimensional.*

The addition theorem for the small dimension ind was given by Toulmin using new operations with transfinite numbers. He gives a simple example of a space for which the addition theorem in usual sense is not true.

B. Levshenko improved Toulmin's results as follows:

L. Theorem 1. *There exist metric compacta R, A, B such that $R = A \cup B$,* $\operatorname{ind} A = \operatorname{ind} B = \omega_0$, $\operatorname{ind} R = \omega_0 + 1$.

L. Theorem 2. *Let R be a metric space and $R = A_1 \cup \ldots \cup A_n$, where A_i are closed; then* $\operatorname{ind} R \leqq \max \operatorname{ind} A_i + \omega_0$.

Let us consider the big transfinite dimension Ind in Čech's sense.

Theorem 1. *If a space R has a big transfinite dimension then R has a small transfinite dimension and* $\operatorname{ind} R \leqq \operatorname{Ind} R$.

Theorem 2. *If a metric space R has big transfinite dimension* Ind R *then* $\operatorname{Ind} R < \omega_1$, *and R is countable-dimensional.*

I have constructed, for every transfinite number $\alpha < \omega_1$, metric compacta I^α for which $\operatorname{Ind} I^\alpha = \alpha$. Levshenko has proved that these compacta I^α may have an arbitrarily high dimension ind.

A space R is called strongly-metrizable when it has an open basis which is a countable union of star-finite coverings.

Theorem 3. *Let a metrizable space R be a countable union of strongly-metrizable subsets R_i; if R has small dimension* ind *then R is countable-dimensional.*

For arbitrary metrizable space this proposition is an unsolved problem.

The proposition inverse to theorem 3 is true for all complete metrizable spaces (completeness is meant in Čech's sense). The following theorem is stronger:

Theorem 4. *Every complete metrizable space R which is an image of a countable-dimensional metric space X by a closed and countable-to-one mapping has small transfinite dimension* ind R.

For the proof one of Nagata's theorems and the Sklyarenko's method are used.

Corollary. *Let R be a countable union of strongly-metrizable subsets and let R have a complete metric. Then the following conditions are equivalent*:

a) *R has small dimension* ind R,

b) *R is countable-dimensional,*

c) *R is an image of a zero-dimensional metric space by a closed and finite-to-one mapping,*

d) *R is an image of a countable-dimensional metric space by a closed and countable-to-one mapping.*

J. Nagata has proved that conditions b) and c) are equivalent generally.

Cali a space R weakly-countable-dimensional when R is a countable union of finite-dimensional closed subsets.

I have constructed a compact metric space which is countable-dimensional but not weakly-countable-dimensional.

Theorem 5. *There exists a universal space for separable metric weakly-countable-dimensional spaces: it is the set of all the points of the Hilbert cube which have only a finite number of non-zero coordinates.*

Recently J. Nagata has constructed a universal space for all metrizable weakly-countable-dimensional spaces with given weight. J. Nagata has proved that the set of all points of the Hilbert cube which have only a finite number of rational coordinates is a universal space for countable-dimensional separable spaces. He also gives some other interesting characterizations of countable-dimensionality.

In his proof of the theorem that the Hilbert cube has no transfinite dimension, W. Hurewicz proved that there exists in this cube a countable number of pairs of closed disjoint sets A_i, B_i with the following property: if the closed sets C_i separate the space between A_i and B_i then the intersection $\bigcap C_i$ is non-void.

The following is a problem of Alexandroff: Let us consider the following property (A) of a space R: for every countable number of pairs of closed disjoint sets A_i, B_i there exist closed sets C_i separating the space R between A_i and B_i with an empty intersection: $\bigcap C_i = \emptyset$.

Alexandroff's problem. *Let R be a compact metric space; is the property* A *equivalent to the property of countable-dimensionality?* For non-compact spaces these properties are not equivalent.

Spaces with property A, called also weakly-infinite-dimensional, have been investigated by Levshenko and Sklyarenko.

L. Theorem 3. *The property* A *is equivalent to the following property* B:

(B) *For every sequence of functions f_i and for every sequence of positive numbers ε_i there exist functions g_i such that $|f_i - g_i| < \varepsilon_i$ and $\bigcap g_i^{-1}(0) = \emptyset$.*

B. Levshenko has generalized to weakly-infinite-dimensional space the addition theorem, the product-theorem, Hurewicz's theorem and others.

S. Theorem 1. *Every strongly-infinite-dimensional compact space contains a Cantor manifold in the following sense:*

The space R is an infinite-dimensional Cantor manifold if it is not cut by any weakly-infinite-dimensional compact subspace.

S. Theorem 2. *Let H be the set of all points of the Hilbert cube which have only a finite number of non-zero coordinates; every compact extension of the space H is strongly-infinite-dimensional.*

Unsolved problems. Let R be a metric space, ind $R = 0$. Has R a big transfinite dimension; is it countable-dimensional; is it weakly-infinite-dimensional?

References

[1] *Б. Т. Левшенко, Е. Г. Скляренко, Ю. М. Смирнов:* О бесконечномерных пространствах. Успехи матем. наук 13, № 5 (*83*), 1958, 203—208.

[2] *Б. Т. Левшенко:* О сильно-бесконечномерных пространствах. Вестн. Моск. ун-та. Сер. физ. мат. и естеств. наук 1959, № 5, 219—227.

[3] *Б. Т. Левшенко:* О бесконечномерных пространствах. Докл. АН СССР, 1961, *139*, № 2, 286—289.

[4] *Е. Г. Скляренко:* О разномерностных свойствах бесконечномерных пространств. Изв. АН СССР Сер. матем. *23*, 1959, № 2, 197—212.

[5] *Е. Г. Скляренко:* О представлении бесконечномерных пространств в виде обратного спектра полиедров. Докл. АН СССР, *134*, № 4, 1960, 773—775.

[6] *Е. Г. Скляренко:* Несколько замечаний о бесконечномерных пространствах. Докл. АН СССР, *126*, № 6, 1959, 1200—1203.

[7] *Ю. М. Смирнов:* 'Об универсальных пространствах для некоторых классов конечномерных пространств. Изв. АН СССР, Сер. матем., *23*, 1959, № 2, 185—196.

[8] *C. H. Toulmin*: Shuffling Ordinals and Transfinite Dimension. Proc. London Math. Soc., *3*, No. 4, 1954, 177—196.

[9] *J. Nagata*: On the Countable Sum of Zero Dimensional Metric Spaces. Fundam. math. 1960, *48*, 1—14.

[10] *J. Nagata*: On a Universal n-Dimensional Set for Metric Spaces. J. reine and angew. Math. *204*, Heft I/4, 1960, 132—138.

[11] *Ю. М. Смирнов:* Несколько замечаний о трансфинитной размерности. Докл. АН СССР *141*, №. 4, 1961, 814—817.