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ON SOME RESULTS CONCERNING TOPOLOGICAL SPACES AND THEIR CONTINUOUS MAPPINGS

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The purpose of this paper is to give a review of some – more or less new – results and problems on the subject. Special attention is paid to the theory of metrization and related questions: in spite of the now classical and definitive metrization theorems of Nagata-Smirnov and Bing the subject is not exhausted and has shown unexpected progress in the very last years.

Theorems concerning the invariance of topological properties under continuous mappings and the representation of topological spaces as images of zero-dimensional spaces are also treated. The last chapter is devoted to some aspects of dimension theory of general spaces.

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There are two general questions which can be roughly formulated as follows:

A. *Which spaces can be represented as images of "nice" (e. g. metric or zero-dimensional, etc.) spaces under "nice" continuous mappings?*

B. *Which spaces can be mapped onto "nice" spaces by "nice" mappings?*

Only continuous mappings will be considered in that what follows: among them there are very different kinds of mappings which are „nice" from different viewpoints: first of all, there are closed and open mappings; next mappings $f: X \leftarrow Y$ may be classified by properties of the counter-images of single points, $f^{-1}y$ $y \in Y$.

Thus we call a mapping *metrizable* if all the $f^{-1}y$ are metrizable spaces. Among them there are *compact metrizable* mappings ($f^{-1}y$ are compacta). Mappings are *bicompact*, if all $f^{-1}y$ are bicompacta. Closed bicompact mappings are called *perfect*. A mapping with bicompact boundaries of the counter-images $f^{-1}y$ is called *peripherally bicompact*, or simply *π-bicompact*. Very interesting are the *S-mappings* (of YU. SMIRNOV and A. H. STONE): these are the mappings whose counter-images $f^{-1}y$ are spaces with countable bases; and so forth.

On the other hand, given an open covering ω of X, one calls a mapping $f: X \rightarrow Y$ an *ω-mapping* if each point $y \in Y$ has a neighbourhood Oy with $f^{-1}Oy$ contained in some element of ω; the notion of a ω-mapping is fundamental in the whole newer development of dimension theory.

It may happen that a mapping of certain type assumes further properties when considered for a restricted class of spaces: a classical example is that every continuous $f: X \to Y$ becomes closed for bicompact X and Hausdorff Y.

Another remarkable case of this kind is discovered by I. A. VAINŠTEIN: a closed $f: X \to Y$ is always π-bicompact if X and Y are metric. This result has been strengthened by A. H. Stone: Given a metric X and a closed $f: X \to Y$; then Y is metrizable if and only if f is π-bicompact.

Next we give some examples of problems of type B.

It has been proved by myself as long ago as in 1924 that every metrizable locally separable space is a sum of mutually disjoint open and closed separable subspaces. Obviously this property is not only necessary but also sufficient for a regular locally separable space to be metrizable. Thus we can say: A locally separable regular space is metrizable if and only if it may be mapped on a metric discrete space by an S-mapping.[1]) YU. SMIRNOV raised the following question: Which metric spaces are S-mappable on a zerodimensional space? Smirnov obtained only a partial answer to this question: namely, he proved that every metric strongly paracompact[2]) space belongs to this category; but there exist non-strongly paracompact spaces which can be mapped by an S-mapping on zero-dimensional metric spaces. On the other hand, if a space allows a closed S-mapping on a zero-dimensional space, it is strongly paracompact; but not all strongly paracompact metric spaces allow such a map.

The following important theorem was essentially proved (although not explicitly stated) by C. H. DOWKER [7], 1948 (and reproved by M. Katětov and V. Ponomarev):

In order that a regular space X be paracompact it is necessary and sufficient that to each open covering ω of X there exists an ω-mapping $f: X \to Y$ of X onto a metric space. The final compact (= Lindelöf) spaces are characterized by assuming Y separable metric in this theorem.

The following theorems of Z. FROLÍK are fundamental in this field:

I. The (completely regular) space X is paracompact and complete (in Čech's sense) if and only if there exists a perfect mapping of X onto a complete metric space.

The second theorem of Z. Frolík belongs to the type A.

II. Let $f: X \to Y$ be closed, X complete metric. The space Y is metric if and only if it is complete (in Čech's sense).

As concerns results of type A, there is the following theorem by V. PONOMAREV [19]:

All spaces with the first Hausdorff axiom of countability and only these spaces are open images of metric spaces.

But, as just proved by A. ARCHANGELSKI [5], a collective normal space which is an image of a metric space under an open bicompact mapping is metrizable.

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[1]) A discrete metric space (and in fact a discrete T_1-space) is a space all of whose points are isolated.

[2]) Strongly paracompact means that every open covering has a star-finite refinement.

There is an example of A. H. Stone of an open compact $f: X \to Y$, where X is metric but Y is not metrizable. V. Ponomarev proved that under these conditions a paracompact Y is always metrizable. V. Ponomarev proved even more: he calls a mapping $f: X \to Y$ of a metric X a *uniform mapping* if for each $y \in Y$ and each neighbourhood Oy, the distance $\varrho(f^{-1}y, X \setminus f^{-1}Oy)$ is positive. Now if X is metric, Y paracompact and $f: X \to Y$ open and uniform, then Y is metrizable.

A. Archangelski proved furthermore [5]:

If X is metrizable, Y is a T_1-space, $f: X \to Y$ is closed and uniform, then Y is metrizable.

A. Archangelski [5] calls a mapping $f: X \to Y$ of a metric X *completely uniform* if to each $y \in Y$ and its neighbourhood Oy, a smaller neighbourhood O_1y can be found in such a way that

$$\varrho(f^{-1}O_1y, X \setminus f^{-1}Oy) > 0 .$$

He settles completely the problem by proving the following theorem:

If X is metric, $f: X \to Y$ open and completely uniform, then the T_1-space Y is metrizable.

The natural question as to which spaces are images of metric spaces under open S-mappings is answered by V. Ponomarev [19], who proved that these spaces and none other have a pointcountable basis.

I considered the condition of existence of a point-countable basis while working on metrization of locally separable (and indeed of locally compact) spaces. I have shown that if this condition is satisfied in a regular locally separable space, then this space is a union of disjoint open and closed separable subspaces and thus is metrizable.

As Yu. Smirnov showed that a locally metrizable space is metrizable if and only if it is paracompact while a separable metric space is even strongly paracompact, it is easily seen that for the metrizability of a regular locally separable space each of the following conditions is necessary and sufficient:

1. paracompactness, 2. strong paracompactness, 3. existence of a point countable basis, 4. existence of a locally countable basis, 5. existence of a star-countable basis, 6. decomposition into disjoint open separable subspaces.

But let us return to spaces with a point countable basis and to their characterization as open S-images of metric spaces.

V. Ponomarev [19] showed that the existence of a point-countable basis is preserved under open S-mappings (while it is obviously not preserved under arbitrary open mappings). The question whether this property is preserved under closed metrizable (or even compact metrizable) mappings remains open.

A. Archangelski and Z. Frolík have proved that a bicompact space which is a closed image of a metric space is metrizable, while A. Miščenko [16] proved recently that every bicompact space with a point-countable basis is metrizable; on the other hand he has constructed a non-metrizable paracompact (normal) space with a point-countable basis. It remains unknown if in this example the assumption of paracompactness can be replaced by final compactness.

Before going further in strengthening the first Hausdorff axiom of countability, let us recall the general metrization theorem by P. URYSOHN and myself [1], proved in 1923 as the first theorem of its kind; today this theorem seems much more natural and satisfactory than it seemed 38 years ago. We called a family $\Sigma = \{\omega_\alpha\}$ of open coverings of given space X *complete* if it has the following property: for each point $x \in X$ and each element $V_\alpha \in \omega_\alpha$ containing this point, the set $\{V_\alpha\}$ so obtained is a basis of the point x in the space X. An alternate formulation of this condition is obviously the following one:

To each $x \in X$ and its neighbourhood Ox there exists in Σ a ω_α such that the star of α in ω_α is contained in Ox.

Our second definition is the following: a covering ω' is a regular refinement of the covering ω, if for each two elements U_1', U_2' of ω' with $U_1' \cap U_2' \neq 0$ there exists an $U \in \omega$ containing $U_1' \cap U_2'$. Obviously the condition of a regular refinement is less restrictive than that of a star-refinement.

The metrization theorem of Urysohn and myself is as follows: A space X is metrizable if and only if there exists in this space a countable complete family of open coverings

$$\omega_1, \omega_2, \ldots, \omega_n, \ldots$$

such that each ω_{n+1} is a regular refinement of ω_n.

One proves easily that in a paracompact regular space the condition concerning regular refinements may be omitted (because of the existence in such a space of star-refinements for any covering). Thus *a necessary and sufficient condition for metrizability of a regular space consists simply in paracompactness and in the existence of a countable complete family of coverings.* (V. Ponomarev).

Remark 1. As first noted by A. Miščenko [16], there exists a regular non-paracompact space in which every covering has a regular refinement.

Remark 2. We say that the space X is symmetrizable if a symmetric function $\varrho(x, x') = \varrho(x', x) \geqq 0$ of two points of X can be defined in such a way that $\varrho(x, x') = 0$ is equivalent with $x = x'$ and $x_0 \in X$ belongs to the closure of a set $M \subseteq X$ if and only if $\inf \varrho(x_0, x) = 0$. We say further that a symmetrziable space is a Cauchy space if it allows a symmetric metric in which each convergent sequence of points $x_n \to x_0$ is a Cauchy sequence (in the sense that $\varrho(x_m, x_n) \to 0$ when $m, n \to \infty$).

A. LUNC [14] has shown that the space of all countable ordinals (with the obvious order topology) is a symmetrizable(!) space but not a symmetrizable Cauchy space. In a joint paper V. NIEMYTZKI and myself [2] have proved that a space X is a symmetrizable Cauchy space if and only if it has a countable complete family of open coverings.

Thus a paracompact symmetrizable Cauchy space is metrizable.

After all these remarks we shall define a property of a topological space which is stronger than the existence of a pointcountable basis. Namely, define a *point-regular basis* as an open basis $\mathfrak{B}$ with the following property:

Any infinite set of elements of $\mathfrak{B}$ containing a given point x is a basis of this point.

It is immediate that every point-regular basis is pointcountable. Moreover, a point-regular basis can also be defined as a basis having the following property:

For each point x and its neighbourhood Ox there is only a finite number of elements of the basis which contain the point x and have points in common with $X \setminus Ox$.

As each element of a point-regular basis is contained in a maximal element of this basis, while any covering by such maximal elements is necessarily point-finite, one can easily conclude weak paracompactness[3]) of spaces having a point-regular basis. Furthermore, if ω_0 is the set of all maximal elements of the given point-regular basis $\mathfrak{B}$, then $\mathfrak{B} \setminus \omega_0$ is again a (point-regular) basis[4]) $\mathfrak{B}_1$, while ω_0 is a point-finite covering of X.

Similarly, the set of all maximal elements of $\mathfrak{B}_1$ is a point-finite covering ω_1 of X and $\mathfrak{B}_1 \setminus \omega$ is a point-regular basis.

In this manner we obtain a sequence

$$\omega_0, \omega_1, \ldots, \omega_n, \ldots$$

of point-finite coverings which is easily seen to be complete. Thus if the space X is paracompact and has a point-regular basis, then X is metrizable, by the Ponomarev version of the theorem of Urysohn and myself [4].

Thus we obtain the following metrization theorem (proved by myself in [1]):

A necessary and sufficient condition for the metrizability of a Hausdorff space is paracompactness combined with the existence of a point-regular basis.

As X is weakly paracompact, paracompactness in this theorem may be replaced by collective normality.

A. Archangelski [6] made a further step in this direction of investigating the metrization problem, and this step is definitive. He calls a basis $\mathfrak{B}$ *regular* if to each point x and to each neighbourhood Ox, a smaller neighbourhood O_1x can be found such that only a finite number of elements of the basis $\mathfrak{B}$ has common with both O_1x and $X \setminus O_1x$.

In the same way as weak paracompactness of a space X follows from the existence of a point-regular basis in X, the ordinary paracompactness of X is a consequence of the existence of a regular basis. As every T_1-space with a regular basis is regular, we have:

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[3]) Weakly paracompact means that every open covering has a point-finite refinement.

[4]) One must be careful — in an obvious manner — with the single point elements virtually present in the basis.

Archangelski's [6] metrization theorem: *In order that a T_1-space be metrizable it is necessary and sufficient that this space have a regular basis.*

2

Until now we have only considered those aspects of the general problems A and B which are more or less connected with metrization and countability. Now let us mention some results and problems concerning the representation of topological spaces as continuous images of zero-dimensional spaces.

I think the first results in this field were some theorems of Hurewicz and myself (proved around 1925); I proved [3] that compacta (i. e. compact metric spaces) are identical with those Hausdorff spaces which are continuous images of the Cantor discontinuum; W. HUREWICZ proved (almost at the same time) his famous characterization of compacta of dimension $\leq n$ as $(n + 1)$-to-1 images of zero-dimensional compacta (or of closed subsets of the Cantor set). Both theorems were the objects of important generalizations until the last years (in particular I have in mind the tremendous generalizations of the results of W. Hurewicz now given by J. NAGATA and E. SKLYARENKO, concerning infinite dimensional compacta).

I proved that every bicompact space of weight τ is a continuous image of a zero-dimensional bicompact space of the same weight, and in fact of a closed subset of the generalized Cantor discontinuum $\mathscr{D}^\tau$ of the same weight τ; the question then arose whether every bicompact space of weight τ is a continuous image of the discontinuum $\mathscr{D}^\tau$ itself. The negative solution of this question is given by E. SZPILRAJN-MARCZEWSKI who also proved some properties of the bicompact spaces now called dyadic, which are such images. E. Marczewski proved that every family of disjoint open subsets of a dyadic bicompact space is at most countable. Further important results are due to N. A. SHANIN [24] and A. ESENIN-VOLPIN. N. A. Shanin proved that no dyadic bicompact space is the sum of a well-ordered increasing family of nowhere dense subsets.[5]) A second theorem by Shanin states that if an ordered bicompact space (with the natural order-topology) is dyadic, then it is necessarily homeomorphic to a compactum lying on the real line.

A. Esenin-Volpin proved that a dyadic bicompact space with first Hausdorff axiom of countability is metrizable.

Thus the dyadicity of a bicompact space is a very strong restriction. On the other hand, L. IVANOVSKI and V. KUSMINOV succeeded in proving the very remarkable theorem that every bicompact topological group (considered as topological space) is a dyadic bicompact space.

In a joint paper appeared in vol. 50 of the Fundamenta Mathematicae, V. Ponomarev and myself have given a characterization of dyadic bicompact spaces and also

[5]) This theorem of Shanin represents a generalization of Baire's theorem on category; for non-dyadic bicompacts it does not hold in general.

of irreducible[6]) dyadic bicompact spaces in terms of coverings. But a characterization of these important spaces by means of more simple and direct set-theoretical properties is still an open problem. Actually we do not know whether a dyadic bicompact space which is of character τ at each point (and thus, according to A. Esenin-Volpin, has weight τ) is the image of $\mathscr{D}^\tau$ under an irreducible mapping.

Now let us pass to general non-compact spaces. To my knowledge the only result in this area is a theorem of Ponomarev [20] asserting that every normal (and in fact every completely regular) space X of weight τ is the image of a certain set $X_0 \subseteq \mathscr{D}^\tau$ under an irreducible perfect mapping.

Of course X_0 is a completely regular space and zero-dimensional in the sense that ind $X_0 = 0$.

The question arises as to when can we suppose dim $X_0 = 0$ in Ponomarev's theorem?

The answer given by Ponomarev is as follows:

Let us call a T_1space X_0 *perfectly zero-dimensional* if each open covering of X_0 has a refinement whose elements are disjoint open (and indeed open-closed) sets; obviously, this property is equivalent to paracompactness combined with dim $X_0 = 0$. Then the following theorem holds:

Among all regular spaces the paracompact ones are characterized by the property that they are perfect images of perfectly zero-dimensional spaces.

The following is an open question:

Is every normal X a perfect image of a normal X_0 with ind $X_0 = 0$?

In concluding this part of my report, it should be emphasized that the problems of types A and B are special cases of the general problem:

Which properties of a space are invariant under multivalued continuous mappings?

In this formulation I understand the continuity of a multivalued mapping in the classical sense of W. Hurewicz which is as a matter of fact, the sense of Cauchy: a multivalued $f: X \to Y$ (where all fx are closed in x) is continuous, if for each neighbourhood Ofx of the closed set fx there exists a neighbourhood Ox of x such that[7]) $fOx \subseteq Ofx$. The inverse mapping f^{-1} sends each point

$$y \in Y \quad \text{into} \quad f^{-1}y = \mathscr{E}(x \in X,\ fx \ni y).$$

A rather detailed theory of multivalued continuous mappings is elaborated by V. Ponomarev in three consecutive papers [21–23].

I will mention here only the following results of these papers.

[6]) An irreducible dyadic bicompact space is the image of $\mathscr{D}^\tau$ under an irreducible continuous mapping. A mapping $f: X \to Y$, $Y = fX$, is called irreducible if there is no closed $A \subset X$, $A \neq X$ with $fA = Y$.

[7]) The image fM of a set $M \subseteq X$ (the "large image") means the set $fM = \bigcup_{x \in M} fx$. If fA is closed for all closed sets A, then f is called closed; if fH is open for all open sets H, f is called open.

A multivalued continued mapping $f: X \to Y$ is said to be *perfect* if it is closed and if for every $x \in X$ and $y \in Y$, the sets fx and $f^{-1}y$ are bicompact.[8]

One of the advantages of the use of multivalued mappings is that the notion of a perfect mapping (like some other important notions) than becomes symmetric: if f is perfect, so is its inverse f^{-1}.

V. Ponomarev proved that all the following properties of a completely regular space are invariant under a perfect multivalued mapping (and thus invariant in both directions, from X to Y and from Y to X):

bicompactness, local bicompactness, paracompactness,
countable paracompactness, completeness in the sense of E. ČECH.

This last invariance is a consequence of one of the extension theorems proved by V. Ponomarev [2] for multivalued continued mappings.

From his four theorems of this kind I shall mention here three.

1st extension theorem.[9] Every closed Y-bicompact mapping $f: X \to Y$ of the T_1-space X onto the T_1-space Y has an extension to a closed mapping ωf of ωX onto ωY: if $\xi = \{A\} \in \omega X$ then

$$\omega f(\xi) = \bigcap_{A \in \xi} [fA]_{\omega Y} .$$

Here ωX, ωY mean, as always, the Wallman extension of X and Y.

V. Ponomarev [21] calls an extension $\varphi : \omega X \to \omega Y$ of a mapping $f: X \to Y$ *bilateral*, if $\varphi^{-1} : \omega Y \to \omega Y$ is an extension of $f^{-1} : Y \to X$.

2nd extension theorem. In order that a mapping of the T_1-space X onto the T_1-space Y have a closed bilateral extension $\varphi : \omega X \to \omega Y$, it is necessary and sufficient that f be perfect. Then ωf is the desired extension and it is the only one which is minimal in the sense that for any closed extension $\varphi : \omega X \to \omega Y$ the inclusion $\omega f(\xi) \subseteq \subseteq \varphi(\xi)$ holds for all $\xi \in \omega X$. For a bilateral extension $\varphi : \omega X \to \omega Y$ we have

$$\varphi(\omega X \setminus X) = \omega Y \setminus Y \quad \text{and} \quad \varphi^{-1}(\omega Y \setminus Y) = \omega X \setminus X .$$

For normal X, Y we have $\omega X = \beta X$, $\omega Y = \beta Y$, and the invariance of the Čech completeness is a consequence of this situation.

The continuity of a multivalued mapping is equivalent to the closedness of its inverse mapping f^{-1}; if f^{-1} is both closed and open, f is called *strongly continuous.*

The **third extension theorem** of Ponomarev is concerned with Y-bicompact strongly continuous mappings of a normal space X onto a normal Y, and asserts that such a mapping f allows precisely one strongly continuous extension $\beta f : \beta X \to \beta Y$;

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[8]) The bicompactness of all $fx \subseteq Y$ is called Y-bicompactness, the bicompactness of all $f^{-1}y \subseteq X$ X-bicompactness of f.

[9]) All mappings are supposed multivalued continuous; brackets mean closure.

this extension is moreover minimal (in the above sense) in the class of all closed extension. If f is open, then βf is also open.[10])

3

1. This last part of my report is devoted to some questions of general dimension theory. Corresponding to the general aim of this paper I shall deal mainly with problems concerning general spaces. But it is impossible not to mention the tremendous progress in the last years of dimension theory of infinite-dimensional spaces which is mainly due to J. NAGATA, YU. SMIRNOV and his pupils B. LEVSHENKO and E. SKLYARENKO.

P. Urysohn was the first to suppose that there are two quite different types of infinite dimensional spaces and particularly of infinite-dimensional compacta (= compact metric spaces): the weakly infinite-dimensional (now universally called the countable dimensional) spaces which can be decomposed into a sum of a countable number of zero-dimensional subspaces, on the other hand those which do not allow such a decomposition. Urysohn formulated the conjecture that the Hilbert cube is strongly infinite-dimensional. Hurewicz proved this conjecture by showing that the Hilbert cube X has the following property:

(A) In X there is a countable number of pairs of closed disjoint sets (A_i, B_i), $i = 1, 2, \ldots$ (i.e. $A_i \cap B_i = \varnothing$) such that whenever closed C_i separate A_i from B_i, the intersection $\bigcap C_i$ is non-void. As no countable-dimension space can have this property (A) I called spaces with this property *essentially infinite-dimensional*; at the same time I formulated the following definition:

We say that a compactum X has the property (A′) if there exists a sequence of n-dimensional cubes Q^n, $n = 1, 2, 3, \ldots$, Q^n a face of Q^{n+1}, and of essential mappings $f_n : X \to Q^n$, such that if π_n^{n+1} denotes the natural projection of the cube Q^{n+1} onto its face Q^n, we have

$$f_n = \pi_n^{n+1} f_{n+1} \, .$$

The definitions bring forward, in a natural manner, these two problems:

1° are the properties (A) and (A′) equivalent for every bicompactum?

2° is anyone of these properties equivalent to the property of a compactum of not having a countable dimension?

B. Levshenko [13] has given a positive answer to the first of these questions; the second remains open.

As concerns the countable-dimensional compacta I will mention only the following result of Nagata-Sklyarenko:

In order that a compactum X not have a countable dimension it is necessary and sufficient that for every mapping f of the Cantor discontinuum C onto X there is at

[10]) This last result was first proved by Ponomarev under a suplementary hypothesis that f is X-bicompact, and thus perfect; A. TAIMANOV [27] showed that this hypothesis may be omitted.

least one point $x \in X$ with an uncountable counter-image $f^{-1}x \subset C$ (which thus has the power of the continuum).

On the other hand, if X has a countable dimension, then there exists a mapping $f: C \to X$ with all counter-images $f^{-1}x$ finite.

It is in my opinion an interesting question to investigate for an X with non-countable dimension, the set X_0 of all points $x \in X$ with an uncountable counter-image $f^{-1}x$. What is the structure of this set? can it be of finite or countable dimension?

I will not dwell any more on infinite-dimensional spaces as there is a rather complete report by Yu. Smirnov on the subject.

2. It is well known that any n-dimensional compactum is the limit space of an n-dimensional projection spectrum (i. e. an inverse spectrum whose elements are simplicial finite complexes). Freudenthal proved that any n-dimensional compactum is the limit space of an n-dimensional polyhedral spectrum (i. e. an inverse spectrum whose elements are polyhedra and whose projections are continuous mappings).

B. Pasynkov [17], [18] and independently S. Mardešić [15] have proved that there are n-dimensional bicompacta which are not limit spaces of n-dimensional polyhedral spectra (although every bicompactum is the limit space of a polyhedral spectrum; but it may be impossible to have in this spectrum projections „onto"). Moreover, B. Pasynkov proved that if an n-dimensional (in the sense $\dim X = n$) bicompactum X is the limit of an n-dimensional polyhedral spectrum with simplicial projections, then necessarily $\operatorname{ind} X = \operatorname{Ind} X = \dim X = n$. Thus the problem of spectral representation of bicompacta is intimately connected with one of the most important problems of dimension theory of general spaces, the problem of interrelations between the different dimensional characteristics of these spaces ($\operatorname{ind} X$, $\operatorname{Ind} X$, $\dim X$). In connection with this problem, B. Pasynkov studied different kinds of spectral approximations; among a number of interesting results (cf. his report to this Symposium) this led to a proof of *the identity*

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X$$

not only for all locally bicompact groups but also for all factor-spaces $X = G/H$ of such a group over a closed subgroup.

I believe this is a very important result indeed. Further progress in the theory of approximation of bicompacta is due to S. Mardešić, who proved that every n-dimensional bicompactum is the limit space of an inverse spectrum whose elements are n-dimensional compact metric space – an unexpected and remarkable result. Mardešič has applied this theorem to obtain another proof of E. Sklyarenko's [22] theorem stating that for every normal space X there exists a bicompact extension bX of the same dimension and weight wbX as X,

$$\dim X = \dim bX\,, \quad wX = wbX\,.$$

The question then arises whether there exists, for all n-dimensional bicompacta (and thus, in virtue of Sklyarenko's theorem, for all n-dimensional normal spaces X)

of weight τ, a universal n-dimensional bicompactum B^n_τ of the same weight τ (universal in the sense that B^n_τ should topologically contain all the n-dimensional X of weight τ).

Let us return to the problem of interrelations between dim X, ind X, Ind X for different spaces X. Obviously ind $X \leqq$ Ind X for all T_1-spaces.

M. KATĚTOV and K. MORITA proved the most important theorem that

$$\dim X = \operatorname{Ind} X$$

for all metric X, while it remains still unknown whether ind $X =$ Ind X for metric spaces. The same identity ind $X =$ Ind X also remains unproved for bicompact spaces: the only known result is dim $X \leqq$ ind X (which I have proved for bicompacta; this has been generalized by Yu. Smirnov and K. Morita to all final compact and even for all strongly paracompact spaces).

It was proved by A. LUNC and O. LOKUCIEWSKI that there exist bicompacta X with dim $X \neq$ ind X, a result which has been strengthened by P. VOPĚNKA in a way which appears exhaustive.

N. VEDENISSOV proved that dim $X \leqq$ Ind X for normal X; it remains unknown whether dim $X \leqq$ ind X holds for paracompact normal spaces.

In a joint paper by V. Ponomarev and myself, questions of this kind were considered from the view-point of families of coverings.

As mentioned previously, a family $\Sigma = \{\alpha\}$ of coverings α of a given space X is called *complete* if to each point $a \in X$ and to each neighbourhood Oa of this point there exists an $\alpha \in \Sigma$ such that the star $S_\alpha a$ of the point a in the covering α is contained in Oa. If we replace in this definition the point a and its neighbourhood Oa by an arbitrary closed set A and its neighbourhood OA, we obtain the definition of a *well-complete* family of coverings.

Finally, the family $\Sigma = \{\alpha\}$ is called *confinally complete* if each open covering ω of X has a refinement $\alpha \in \Sigma$.

Another important definition is the following:

We shall say that the closed covering α' is a strong refinement of α, if α' is a refinement of α and if each element of α is the union of all elements of α' contained in it,

$$A = \bigcup_{\substack{A' \in \alpha' \\ A' \subseteq A}} A' .$$

We can consider the relation of strong refinement as an ordering relation in the system $\Sigma = \{\alpha\}$ of closed coverings. In particular, $\Sigma = \{\alpha\}$ is directed if any two coverings $\alpha_i \in \Sigma$, $\alpha_2 \in \Sigma$ have a common strong refinement $\alpha \in \Sigma$.

Now V. Ponomarev and myself [3] proved the following theorems:

Theorem I. *If in the space X there exists a directed complete (resp. well-complete) family Σ of closed covering α, each of order $\leqq n + 1$, then for the space X (which in this case is obviously normal) there hold the relations* ind $X \leqq n$ (or Ind $X \leqq n$ *respectively*), *and* (*obviously*) dim $X \leqq n$.

In this case for any subspace $B \subseteq X$ which is the intersection of some $p + 1$ elements of a fixed $\alpha \in \Sigma$ there exists a directed complete (or well-complete respectively) family of closed coverings β, each of order $\leqq n - p + 1$, such that $\operatorname{ind} B \leqq \leqq n - p$ ($\operatorname{Ind} B \leqq n - p$ respectively).

In this theorem the coverings $\alpha \in \Sigma$ can be supposed finite as well as locally finite.

Next we confine ourselves to bicompact normal spaces. In this case "complete" means confinally complete.

We call a bicompactum X with $\dim X = n$ perfectly n-dimensional, if in X there exists a directed complete family of finite closed coverings. In this case we have by theorem I

$$\dim X = \operatorname{ind} X = \operatorname{Ind} X .$$

Now any n-dimensional bicompactum X (in the sense $\dim X = n$) has complete systems Σ of closed (even of closed canonical[11]) coverings of order $n + 1$. Now it follows from theorem I that in the case of $\dim X \neq \operatorname{ind} X$ none of these families of coverings can be directed. This means that if we direct the given family Σ (say, of canonical coverings α of order $n + 1$) by adding new canonical coverings, then these new coverings necessarily fail to be of order $\leqq n + 1$. This negative result seems to be the most interesting consequence of theorem I.

Theorem II. *Every perfectly n-dimensional bicompactum is the image of a zero-dimensional bicompactum under an $(n + 1)$-to-1 continuous mapping.*

On the other hand, every bicompactum X which is the image of a zero-dimensional one under an $(n + 1)$-to-1 mapping has a complete family of (even canonical) coverings of order $n + 1$ and therefore has $\dim X \leqq n$; thus

Theorem III. *Among all n-dimensional bicompacta, the perfectly n-dimensional and only these are $(n + 1)$-to-1 images of zero-dimensional bicompacta.*

It follows that a perfect n-dimensional bicompactum is not an image of a zero-dimensional bicompactum under a $(1, k)$-mapping with $k < n + 1$.

It is of course possible to give a suitable generalization of these results to the paracompact case (for theorem I this generalization is immediate).

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