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Non-F-spaces

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NON-F-SPACES

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Praha

I. In this note some theorems about topological spaces, in which the closure of a set is not always a closed set, are shown. The topology u on a set P is a mapping, which assigns a set $uM \subset P$ to every set $M \subset P$ and satisfies the following axioms: $u\emptyset = \emptyset$, $u(X) = (X)$, $u(M_1 \cup M_2) = uM_1 \cup uM_2$. The condition $u(uM) = uM$, called axiom F, is not required in general; thus we distinguish among F-spaces (i. e. spaces, satisfying F-axiom) and non-F-spaces. Non-F-spaces were called "gestufte Räume" by F. Hausdorff. In non-F-spaces neighbourhoods of sets and interiors of sets are defined as follows: a set U is a neighbourhood of a set M if $M \cap u(P - U) = \emptyset$; Int $M = \{x \in M; M$ is neighbourhood of $x\}$. The non-F-topology problems have been dealt with by some Czech mathematicians. E. Čech defined on a topological space (P, u) a new topology $\tilde{u}$, called the F-reduction of u such that $\tilde{u}$ satisfies axiom F and $\{P - uM;\ M \subset P\}$ is an open base for it. Therefore the neighbourhoods of points in $(P, \tilde{u})$ are interiors of neighbourhoods in (P, u). Evidently, $\tilde{u}$ is finer than u and the equality $u = \tilde{u}$ holds if and only if u is an F-topology.

Theorem 1. *Let (P, v) be an F-space. Then there exists a non-F-topology u on P such that $\tilde{u} = v$ if and only if v is not maximal (i. e. there exists an infinite set $M \subset P$ such that $vM \neq P$).*

Proof in [17].

II. For the greater part of non-artificially constructed non-F-spaces the F-reduction is discrete. It refers especially to the spaces of real functions with a topology, defined by means of convergence of sequences of functions at each point. Let $D(Q)$ denote some set of real function on an F-space Q; we say that a sequence $\{f_n\}$ of points of $D(Q)$ converges to $f \in D(Q)$ if $\lim_{n \to \infty} f_n(x) = f(x)$ for all $x \in Q$. On $D(Q)$ the topology u is defined in such a way that the closure uM of $M \subset D(Q)$ is the set of all limit-points of all sequences of points of M. This topology u shall be considered on a set of all (or all bounded) real continuous functions on some F-space Q (denoted by $C(Q)$) and on a set of all characteristic functions on some set Q (denoted by $\chi(Q)$). We recall that a space $(D(Q), \tilde{u})$ is discrete if and only if for every $f \in D(Q)$ there exists $H_f \subset D(Q)$ such that every $g \in D(Q)$, $g \neq f$ is a limit point of some sequence of points of H_f, but no sequence of points of H_f converges to f (this follows immediately from the definition of the F-reduction of a topology).

Theorem 2. *Let Q contain a countable dense subset. Then $(C(Q), \tilde{u})$ is discrete if and only if $(C(Q), u)$ is a non-F-space.*

Proof in [15].

For non-separable Q this theorem does not hold. There exists even a compact Hausdorff space Q, for which $(C(Q), u)$ is a non-F-space but $(C(Q), \tilde{u})$ is not discrete. The example of this space is contained in [15].

Theorem 3. *Let Q be a normal space, containing a discrete, normally imbedded subset,*[1]) *the power of which is $\aleph = \aleph^{\aleph_0}$ ($\aleph$ denotes an arbitrary cardinal number) and a dense subset, the power of which is $\leqq 2^{\aleph}$. Then $(C(Q), \tilde{u})$ is discrete.*

Proof in [15].

Theorem 4. *Let Q contain a countable dense metrizable subset; let every point of Q have a complete collection of neighbourhoods such that each neighbourhood from this collection is dense-in-itself, normal, non-meager space. Let $R \subset C(Q)$ be a ring of functions such that*

(a) $uR = C(Q)$,

(b) *if $A \subset Q$ is closed, $x \in Q$, $x \notin A$, then there exists $f \in R$ such that $f(x) = 0$, $f(y) = 1$ for all $y \in A$.*

Then for every $f \in C(Q)$ there exists $H_f \subset R$ such that $uH_f = C(Q) - (f)$.

Proof in [18].

III. Theorem 2 leads to one problem (which as far as I know, has not yet been fully solved) when $(C(Q), u)$ is an F-space and when it is not. Two theorems follow concerning this:

Theorem 5. *If for every $f \in C(Q)$ the set $f(Q)$ is countable, then $(C(Q), u)$ is an F-space.*

This theorem follows immediately from the following proposition.

Proposition 1. If for every $f \in C(Q)$ the set $f(Q)$ is countable, then also a set $g(Q)$ is countable, where g is any continuous mapping on Q in a separable metric space.

Proofs of this proposition 1 and of the theorem 5 are in [18].

Theorem 6. *If there exists $f \in C(Q)$ such that $f(Q)$ contains a dense-in-itself, non-meager closed part, then $(C(Q), u)$ is a non-F-space.*

Proof in [18].

IV. Definition. Let N be the set of all natural numbers, $\alpha, \beta \in N^N$. We write $\alpha \prec \beta$ if $\alpha(x) < \beta(x)$ for all $x \in N$ except a finite number.

Let ϱ be the smallest power of an unbounded chain in this order of N^N.

We call a set $M \subset N^N$ a hereditary unbounded system if for every infinite $A \subset N$ and every $\alpha \in N^N$ there exists $\beta \in M$ such that $\alpha(x) < \beta(x)$ for an infinite number of $x \in A$.

Let τ be the smallest power of a hereditary unbounded system.

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[1]) I. e. every bounded real continuous function can be extended on the whole Q.

Theorem 7. *Let Q be a set. Then $(\chi(Q), u)$ is a non-F-space if and only if* card $Q \geqq \tau$.

Proof in [18].

V. Let (P, u) be a space, u^* the F-topology, which we get from a topology u by iterating the closure operator. Following E. Čech we call this topology the F-modification of u. Consequently, the F-modification u^* of u is the finest F-topology from all F-topologies, coarser than u.

Theorem 8. *Let (P, v) be an F-space. Then v is not an F-modification of any non-F-topology on P if and only if v satisfies the condition $\mathscr{D}_x$ for every $x \in P$.*

Condition $\mathscr{D}_x$: If $A \subset P$, $x \in vA - A$, then there exists $B \subset A$ such that $x \in vB$, $x \notin v(vB -' A - (x))$.

This condition has a very simple form for regular spaces: If $A \subset P$, $x \in vA - A$, then there exists $B \subset A$ such that $vB - B = (x)$.

The proof of theorem 8 is contained in [17].

Such an F-space, the topology of which is an F-modification of no non-F-topology, is called a strong F-space. Immediately from theorem 8 it follows that every metric space is a strong F-space. The product of an uncountable number of intervals $\langle 0,1 \rangle$ is not a strong F-space. This is implied by the following theorems:

Theorem 9. *If (P, v) is a product of an uncountable number of F-space, each of which contains at least two points, then there exists an uncountable number of topologies u on P such that $u^* = v$, the order*[2]) *of u is 2 and for every $x \in P$ there exists $H_x \subset P$ such that $x \in u(uH_x) - uH_x$.*

Theorem 10. *Let (P_λ, v_λ) $(\lambda \in \Lambda)$ be F-spaces, satisfying the first axiom of countability. Let* $2 \leqq$ card $P_\lambda \leqq$ card $\Lambda > \aleph_0$. *Let (P, v) be the product of the spaces (P_λ, v_λ). Then there exist $2^{\text{card}\,P}$ different topologies u on P such that $u^* = v$, $\tilde{u}$ is discrete and the order of u is 2.*

If (P, v) is a product of F-spaces (P_λ, v_λ), $\lambda \in \Lambda$, $2 \leqq$ card $P_\lambda \leqq$ card $\Lambda > \aleph_0$, then there exists [17] a disjoint system $\{A_x;\ x \in P\}$ of dense subsets of P and such that if all (P_λ, v_λ) satisfy the first axiom of countability, then every A_x satisfies (a) from the following proposition:

Proposition 2. Let (P, v) be an F-space. Let there exist the collection $\{A_x;\ x \in P\}$ of subsets of P such that

(a) $x \in vA_x - A_x$ and if $B \subset A_x$, $x \in vB$, then $x \in v(vB - A_x - (x))$,

(b) for $y \in vA_x - A_x - (x)$ there exists a neighbourhood Y of y such that $Y \cap A_x \cap A_y = \emptyset$.

Then there exists a topology u on P, the order of which is 2, $u^* = v$ and for every $x \in P$ there exists $H_x \subset P$ such that $x \in u(uH_x) - uH_x$.

If in addition,

[2]) If we define for $M \subset P : u^1 M = uM$, $u^\beta M = u(\bigcup_{\gamma < \beta} u^\gamma M)$, then the order of topology u is the smallest ordinal number α such that $u^{\alpha+1} M = u^\alpha M$ for all $M \subset P$.

(c) $vA_x = P$ for every $x \in P$
then $\tilde{u}$ is discrete.

The proofs of proposition 2, theorems 9 and 10 are contained in [17].

VI. In this section the T_1-axiom for spaces is not assumed. If (Q, v) is an F-space and f its mapping onto a set P, usually a quotient-topology on P is defined as a finest F-topology for which f is a continuous mapping. If we substitute the word "F-topology" in this definition through "topology" only, we get a new notion of the quotient-topology. Evidently the "old-quotient-topology" is the F-modification of the "new quotient-topology".[3])

Theorem 11. *If (P, u) is a space, then there exists an F-space (Q, v) and a mapping f of (Q, v) onto P such that (P, u) is a quotient space ("new" of course). It is possible to choose $Q \supset P$ and such that the subspace $Q - P \subset\subset (Q, v)$ is discrete and the subspace $P \subset\subset (Q, v)$ is homeomorphic with $(P, \tilde{u})$.*

Proof in [17].

VII. Three theorems follow about F-modification of topology u (defined by means of convergence of functions at each point) on a set $\chi(Q)$ of all characteristic functions on some set Q:

Theorem 12. *The following statements are equivalent:*

(a) $(\chi(Q), u)$ *is not regular,*[4])

(b) $(\chi(Q), u^*)$ *is not regular,*

(c) card $Q \geqq \aleph_1$.

Proof in [18].

Theorem 13. *Let* card $Q \geqq \varrho$ *(c. f. IV). Then for every $f \in \chi(Q)$ there exists a countable set $A \subset \chi(Q)$ and a closed subset T of $(\chi(Q), u)$ such that $f \notin T$ and if U is a neighbourhood of f in $(\chi(Q), u)$, then $T \cap u(U \cap A) \neq \emptyset$.*

Proof in [15].

Let σ be the smallest power of a system $\mathscr{A}$ of subsets of some countable infinite set A such that if $B \subset A$ is infinite, then there exists $C \in \mathscr{A}$ such that $B - C$ and $B \cap C$ are infinite.

Theorem 14. *The space $(\chi(Q), u^*)$ is countably compact if and only if* card $Q < < \sigma$.

Proof in [18].

VIII. Finally, I would like to give a summary of directions of recent non-F-topology research. We could roughly divide papers about non-F-topology into three groups. The first group is composed of studies of properties of a set of ordinal numbers, which we get by iterating the closure-operator (papers [4], [5], [9]); the second group is a study of topologies, defined by means of various convergences of sequences,

[3]) This definition has been communicated to me by M. KATĚTOV.

[4]) The definition of regularity is the same for non-F-spaces as well as for F-spaces; if U is a neighbourhood of x, then there exists a neighbourhood of x, the closure of which is contained in U.

especially convergences of sequences of functions (papers [3], [6], [7], [8], [11], [12], [15], [16], [18]) and last but not least are studies of "pure theory" of non-F-spaces (papers [1], [2], [3], [10], [13], [14], [17]).

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