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CIRCUMSCRIBING CONVEX SETS

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In the main this note considers the fitting of cubes in cubes. Details of the proofs and applications are reserved for other publication. The genesis of the primary problem is the Kakutani theorem that a convex body K in R^3 admits a circumscribing cube. Recently one of us generalized this result to an assertion which in its weakest form guarantees that there are a non finite number of such cubes [1]. If, however, the edge length of the circumscribing cube is prescribed, it appears plausible that for some K there are at most a finite number of such cubes. Indeed if K is a cube, this is the fact as we show below. Other results in this range of ideas are included.

We shall use the following conventions: C_0 is a fixed cube with vertices $(\pm 1, \pm 1, \pm 1)$ and C is the cube of edge length $2a$. The *long diagonals* are those connecting antipodal vertices. C_0 *circumscribes* C if every face of C_0 contains at least one vertex of C, so only 6 vertices of C need touch faces of C_0. A rhombord has congruent rhombus faces.

Lemma 1.1. *If C_0 circumscribes C, then the center of C must be at the origin.* Fairly direct geometric arguments show that

Theorem 1.2. *If $C \neq C_0$ it is impossible that all 8 of the vertices of C lie on faces of C_0.*

Let C_1 be a cube of edge length $2a$, center 0 and faces parallel to those of C_0. The key result is

Theorem 2.1. *If C_0 circumscribes C, then C can be obtained by rotating C_1 about a long diagonal of C_0 and the side length of C is at least $2a = 6/5$.*

An aesthetically satisfying proof starts with the rotation matrix A. The vectors $(a, -a, a)$, $(a, a, -1)$ and $(-a, a, a)$ are taken into vectors with end points on $x = 1$, $y = 1$ and $z = 1$ respectively. Then with λ the unit vector along the axis of rotation and θ the angle of rotation, choice of $1 - \cos\theta$, $\sin\theta$ and $a^{-1} - \cos\theta$ as variables leads, with a minimum of manipulation, to the conclusion $\lambda_1 = \lambda_2 = \lambda_3$.

A *frame* F consists of 3 equally spaced radii with the end points on the planes $x = 1$, $y = 1$ and $z = 1$ respectively. The angle α between any pair is called the *face angle*. The frame is *admissible* if the end points lie inside the square faces of the cube. The inscribed cube is associated with a frame of face angle $\alpha_0 = 2 \operatorname{arc\,sin} 1/\sqrt{6}$.

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The problem of existence of admissible frames can also be formulated as follows: Let S_1, S_2, S_3 be three congruent circles on the sphere of radius R, with centers on the vertical and on the two horizontal coordinate axes. Let Λ be an axis of rotation. For what points of S_1 do rotations of amounts $2\pi/3$ and $4\pi/3$ yield points on S_2 and on S_3 respectively? The radii to these point triples constitute frames.

Evidently for $\Lambda = \Lambda_0$ the axis through 1, 1, 1, every point of S_1 has the required property and for each value of R there are two admissible orthogonal frames F_0. Consistently with [1] it is clear that the set of F_0 frames of varying lengths consitutes a continuum.

Let e_1, e_2, e_3 be the vectors of a frame and let $e_0 = e_1 + e_2 + e_3$.

Lemma 3.1. $\{\pm e_i \mid i = 0, \ldots, 3\}$ *are the vertices of a rhombord and if* $\alpha = \alpha_0$ *this rhombord is a cube.*

It does not follow from the existence of an admissible triple that the associated rhombord is circumscribed by C_0 since it is essential that the diagonal $[-e_0, e_0]$ of the rhombord be at most the length of the long diagonals of C_0. If x_1, y_1, 1 are the coordinates of one vertex of an admissible frame, then for $\Lambda = \Lambda_0$ the condition mentioned requires that $x_1 + y_1 \leqq 0$. For a thin enough rhombord K with long diagonal coinciding with that of C_0, arbitrary rotation of K about this diagonal yields a circumscription. However, only two of the rhombord vertices touch C_0 here. We therefore bar this type of situation by insisting that circumscription imply that the vertices touch away from the vertices of C_0.

The special property of $\Lambda = \Lambda_0$ is evidenced by the following 2 theorems.

Theorem 4.1. *For a small enough deleted neighborhood of* Λ_0 *each* Λ *determines a unique admissible frame.*

Remark. It is easy to give examples of admissible frames with Λ "far" from Λ_0, for instance through 1, 2, 2.

Theorem 4.2. *For no deleted neighborhood of* Λ_0 *is the totality of rhombords associated with admissible frames circumscribed by* C_0.

Of course a circumscribed convex set is obtained from each frame in Theorem 4.1 by taking the convex hull of the admissible frame vectors and their antipodals. By a *semi regular octahedron* we understand one with two parallel equilateral triangles as faces and 6 other congruent isosceles triangle faces. Thus

Corollary 4.3. *For each* A *in a small enough deleted neighborhood of* A_0 *there is a unique semi regular octahedron inscribed in* C_0 *with* Λ *orthogonal to the equilateral faces.*

On dualizing and noting the dual of C_0 is a regular octahedron and that of the semi regular octahedron is a rhombord

Theorem 5.1. *Each rhombord circumscribes at most one regular octahedron of fixed size up to obvious symmetries.*

Bibliography

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