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POINT COUNTABLE OPEN COVERINGS IN COUNTABLY COMPACT SPACES

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The aim of what follows essentially is that of remarking some very simple properties of countably compact topological spaces which seem to have escaped notice previously and may be regarded as to belong to the folklore of the subject.

For a matter of convenience, the following property of a topological space will be used throughout:

(*) *For any discrete*[1]) *family*

$$(F_\lambda)_{\lambda \in L}$$

of non empty closed sets of the topological space the index set L is necessarily countable.

Obviously, if the topological space X fulfils condition (*) then any closed subspace also does. In addition, if the topological space X is the countable union of a sequence of its closed subspaces verifying property (*), then X likewise fulfils property (*).

For instance, the above property (*) holds in each of the following cases.

1) *X is countably compact.* As a matter of fact, in this case, the index set L turns out to be finite.

2) *X is the countable union of a sequence of its closed countably compact subspaces.*

3) *X is a collectionwise normal space satisfying the countable chain condition.*

4) *X is a Lindelöf space.*

The following lemma will play a fundamental role in the sequel.

Lemma. *If the topological space E fulfils property* (*), *then for each point countable*[2]) *open convering $(U_\iota)_{\iota \in I}$ of E there exists a countable subcovering $(U_\iota)_{\iota \in H}$ (H being a countable subset of I).*

Proof. Let

$$V = \bigcup_{\iota \in I} (U_\iota \times U_\iota) .$$

[1]) It should be emphasized that $(F_\lambda)_{\lambda \in L}$ is discrete, according to the usual meaning, if for each point x of X there exists a neighbourhood U of x such that the set $\{\lambda \in L \mid F_\lambda \cap U \neq \emptyset\}$ contains one element at the most.

[2]) In the sense that, for each point x of E the subset $\{\iota \in I \mid x \in U_\iota\}$ of I is countable.

Evidently, V is an open symmetric neighbourhood of the diagonal Δ_E of $E \times E$ and, therefore, in force of a known lemma[3]), there exists a subset A of E such that[4])

$$E = V(A) \left(= \bigcup_{x \in A} V(x)\right) \tag{1}$$

and such that $x \in A$, $y \in A$, $x \neq y$ implies $y \in \complement_E(V(x))$.

Notice now that $V(x)$ is open for any $x \in E$ and that

$$V(x) = \bigcup_{\iota \in I_x^*} U_\iota \tag{2}$$

where we assume that $I_x^* = \{\iota \in I \mid x \in U_\iota\}$.

If $x \in A$ one has:

$$x \in \bigcap_{y \in A - \{x\}} \complement_E(V(y)) = \complement_E\left(\bigcup_{y \in A - \{x\}} V(y)\right)$$

hence, $\bigcup_{y \in A - \{x\}} V(y)$ being open, one gets

$$\{\bar{x}\} \subset \complement_E\left(\bigcup_{y \in A - \{x\}} V(y)\right) \subset V(x)$$

while, on the other hand, from $y \in A - \{x\}$, it follows that

$$\{\bar{x}\} \cap V(y) = \emptyset$$

and so,

$$(V(x))_{x \in A}$$

being an open cover of E, one infers that

$$(\{\bar{x}\})_{x \in A}$$

is a discrete family of non empty closed subsets of the space E.

As a consequence of property (*), it turns out that the set A is countable and therefore, letting

$$H = \bigcup_{x \in A} I_x^*,$$

each I_x^* being likewise countable, it turns out that H itself is countable.

As, in force of (1) and (2), one gets

$$E = \bigcup_{x \in A}\left(\bigcup_{\iota \in I_x^*} U_\iota\right) = \bigcup_{\iota \in H} U_\iota,$$

the lemma is proved.

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[3]) The lemma referred to in text can be found, for instance, in [1].

[4]) If V is a subset of the cartesian product $Y \times Z$ of the two sets Y and Z, then for each subset A of Y we assume

$$V(A) = \{z \in Z \mid \exists y \in A : (y, z) \in V\}.$$

For $y \in Y$ one puts $V(y) = V(\{y\})$.

Once that this lemma has been established, the following proposition trivially holds:

Proposition 1. *If X is a countably compact topological space then for each point countable open covering of X there exists a finite subcovering.*

At this point, the proposition proved and a result of A. MIŠČENKO [4] enable us to affirm that:

Proposition 2. *If X is a (T_1) topological space, then the following two proposition are equivalent*:

a) *X is countably compact and possesses a point countable base*

b) *X is compact and possesses a countable base.*

This result is already known[5]) (H. H. Corson and E. A. Michael [3]) and when, in it, X is more particularly assumed to be (T_1) regular, then proposition b) above may be replaced by:

b)′ *X is a metrizable compact space.*

Another application of Proposition 1 follows, after stating the definition.

Definition 1. *The topological space X is called countably metacompact if for each open covering of X there exists a point countable open refinement.*

Obviously one has

Proposition 3. *If a countably metacompact space X is countably compact then it is also compact.*

This result slightly generalizes a very well known result by ARENS and DUGUNDJI [2] replacing the metacompactness assumption by countable metacompactness and dropping separation axioms.

The present paper is a slightly modified version of [1].

References

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[2] *R. Arens and J. Dugundji:* Remarks on the concept of compactness. Portug. Math. *9* (1950), 141—143.

[3] *H. H. Corson and E. A. Michael:* Metrizability of certain countable unions. Illinois J. Math. *8* (1964), No. 2, 351—360.

[4] *A. Miščenko:* Spaces with point countable bases. Doklady Acad. Nauk, URSS, translations of the American Math. Soc., *3* (1962), No. 3, 855—858.

[5]) While attending the Second Prague Symposium on General Topology, the writer has been informed by A. ARHANGELSKIJ that a generalization of Proposition 2 above, has been published by him and V. V. PROIZVOLOV in *Vestnik Moskovscogo Universiteta* (1966).