William Wistar Comfort

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In: (ed.): General Topology and its Relations to Modern Analysis and Algebra, Proceedings of the second Prague topological symposium, 1966. Academia Publishing House of the Czechoslovak Academy of Sciences, Praha, 1967. pp. 95--100.

Persistent URL: http://dml.cz/dmlcz/700878

LOCALLY COMPACT REALCOMPACTIFICATIONS

W. W. COMFORT[1])

Amherst

I'd like to begin this talk by citing a theorem which will appear in a paper I co-authored recently with Stelios Negrepontis. I hope and believe that the theorem has, as they say, merit and interest in its own right, but at this moment I do not want to dwell at all upon the theorem itself and its proof, but rather to deduce from it a number of consequences. As you will notice, each of these consequences has among its hypotheses one of the two following somewhat artificial conditions: either "υX is locally compact" or "υX is a k-space". This talk is to be devoted chiefly to a study of the first of these two conditions.

(Incidentally, perhaps it is reasonable to remind you of the main properties enjoyed by υX, the Hewitt realcompactification of the completely regular Hausdorff space X. It is realcompact, of course – i.e., homeomorphic to a closed subset of a product of lines – and roughly speaking it does for the ring $C(X)$ of real-valued continuous functions on X about what the Stone-Čech compactification βX does for the ring $C^*(X)$ of bounded elements of $C(X)$. Let me be more specific. Each continuous function mapping X into the real line, or in fact into any realcompact space whatever, admits a continuous extension mapping υX into that same realcompact space. And υX is the only such realcompact space containing X densely. As Gillman and Jerison remind us gently in their remarkable work [3], the symbol υ is the Greek upsilon, not nu. In a letter dated 24 May 1966, Edwin Hewitt, reflecting back on his original paper [6], writes "I chose upsilon by some crude association with the word 'unbounded', just as Čech probably chose 'β' because he was thinking of bounded functions." If Hewitt's conjecture is correct and the word "bounded" was paramount in Čech's mind when he embedded X in a product of bounded intervals indexed by bounded continuous real-valued functions on X, one might reasonably wonder today whether it did not also occur to Čech, first as a sort of play on words and then as a subject for serious speculation, to embed X in a product of unbounded intervals indexed by unbounded continuous real-valued functions on X. Given the premise I believe the conclusion to be more than likely, but so far as I am aware there is no concrete evidence to suggest that Čech was at any time on the verge of constructing

[1]) The author gratefully acknowledges support from the National Science Foundation under contract NSF GP-5750. He also thanks the Research Council of the University of Massachusetts for support received from Faculty Research Grant FR-J37-67(1).

the space which we know today as the Hewitt realcompactification. My own guess, based solely on a perusal of Čech's paper [1], is that the symbol β, if indeed it had any particular significance in Čech's mind, properly stands not for the word "bounded" but for the word "bicompact".)

I beg your pardon for that unexpectedly lengthy parenthetical digression, and I now return to that Negrepontis theorem I promised you. It asserts, or at least one of its easy consequences asserts, that *if* Y *is a compact space of nonmeasurable cardinal, then* $\upsilon(X \times Y) = \upsilon X \times Y$ *for every space* X. A reformulation looks like this: *If* Y *is a compact space of nonmeasurable cardinal, then for each space* X *every function in* $C(X \times Y)$ *is the restriction to* $X \times Y$ *of some function in* $C(\upsilon X \times Y)$. If the cardinality hypothesis is omitted and measurable cardinals exist, by the way, then the resulting assertion is false. Since any compact space is its own Hewitt realcompactification, the theorem I just cited may be viewed as furnishing a criterion sufficient for the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$, and it is in this setting that I wish you would consider it. It is not the only such theorem known. The doctoral dissertation [5] of Anthony W. Hager contains an interesting result of the same form, based on this beautiful theorem of Glicksberg [4] which was reproved very elegantly by Frolík in [2]: In order that $\beta(X \times Y) = \beta X \times \beta Y$, where X and Y are infinite spaces, it is necessary and sufficient that $X \times Y$ be pseudocompact. This Glicksberg-Frolík theorem is, as a matter of fact, the genesis of the paper I am discussing with you now. Seeking the "υ" analogue of that "β" theorem, we would like to have conditions on X and Y necessary and sufficient that the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ be valid. We are considering throughout only completely regular Hausdorff spaces.

To avoid senseless repetition and in the interest of simplicity, I'll assume now that any space which arises in future discussion is of nonmeasurable cardinal. Suppose now that Y, instead of being compact, is assumed simply to be locally compact. And let $f \in C(X \times Y)$. According to the Negrepontis theorem I just quoted, each point y of Y admits a (compact) neighborhood U with the property that the restriction of the function f to the space $X \times U$ extends continuously to $\upsilon X \times U$. Using this observation we can construct a well-defined function on all of $\upsilon X \times Y$ which agrees with f on $X \times Y$ and which is locally continuous. Of course a locally continuous function is continuous, so our construction actually yields the following result.

Corollary. *If* Y *is locally compact and realcompact, then* $\upsilon(X \times Y) = \upsilon X \times Y = \upsilon X \times \upsilon Y$ *for each space* X.

Suppose now that Y is a k-space, so that in particular each real-valued function on Y continuous on each compact subset of Y is automatically continuous on Y. Then $Y \times Z$ is a k-space for any locally compact space Z, and so we have: If Y is a k-space and υX is locally compact, then $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$. (You can prove that theorem for yourself by applying the corollary above two times, firstly to sets of the form $X \times U$ where U is compact in Y, and secondly to the space $Y \times \upsilon X$, which now satisfies the hypotheses imposed in the corollary upon $X \times Y$.)

The same sort of interchange allows us to prove the following result, where as before both X and Y are subject to our standing nonmeasurability hypothesis.

Corollary. *If* $\upsilon X \times Y$ *and* $\upsilon X \times \upsilon Y$ *are both k-spaces, then* $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$.

I hope that what I have said so far suffices to convince you that any serious study of the relation $\upsilon(X \times Y) = \upsilon X \times \upsilon Y$ is likely to involve a study of the property "υX is locally compact".

By way of initiating this study I want to present a new proof of a result which I believe should be called a Hager-Johnson theorem. The theorem appears in Hager's thesis [5], with a lengthy, ingenious, indirect proof concocted by D. G. Johnson. The proof I want to show you now has no advantage over that of [5], except that it is more simple and more direct.

Theorem (Hager-Johnson). *If* U *is open in* X *and* $\mathrm{cl}_{\upsilon X}\, U$ *is compact, then* $\mathrm{cl}_X\, U$ *is pseudocompact.*

Proof. Hoping to achieve a contradiction, let us suppose that some continuous real-valued function f with domain $\mathrm{cl}_X\, U$ is unbounded. Since f is unbounded on U, we can, beginning with any point x_1 in U, construct inductively a sequence of points x_n in U for which $|f(x_{n+1})| > |f(x_n)| + 1$. Now for each n let g_n be a continuous function on X for which $g_n(x_n) = n$ and for which $g_n(x) = 0$ whenever $|f(x_n) - f(x)| \geqq \frac{1}{4}$. It is clear that each point of X admits a neighborhood throughout which every one of the functions g_n, with at most one exception, vanishes identically. (Specifically, one can choose for a neighborhood of x the set $f^{-1}((f(x) - \frac{1}{4}, f(x) + \frac{1}{4}))$.) Therefore the function g defined on X by the relation $g = \sum_{n=1}^{\infty} g_n$ is continuous. Being an element of $C(X)$, g extends continuously to υX. But this extension is, like g itself, unbounded on U. Hence it is unbounded on the compact space $\mathrm{cl}_{\upsilon X}\, U$. We have achieved the desired contradiction.

This theorem of Hager-Johnson allows us to describe a relation between the local compactness of υX and the local pseudocompactness of X itself. The concept of local pseudocompactness has, so far as I am aware, been avoided by mathematicians, and I regret the necessity of introducing it to you now. I shall use the term to mean just what it ought to mean.

Definition. A space is *locally pseudocompact at its point* x if there is a local neighborhood base at x consisting of pseudocompact sets.

Since the closure of an open subset of a pseudocompact space is pseudocompact, and since the completely regular Hausdorff spaces we are considering are surely regular, a space is locally pseudocompact at the point x if and only if x admits a pseudocompact neighborhood. If a space is locally pseudocompact at each of its points, I shall say simply that the space is locally pseudocompact.

Theorem. *In order that X be locally pseudocompact, it is necessary and sufficient that there exist a locally compact space Y for which $X \subset Y \subset \upsilon X$.*

Proof. The sufficiency is easy, because if Y is as hypothesized and x is a point of X, then some neighborhood of x in Y (let's call it K) is compact. The set $\mathrm{int}_X(K \cap X)$ is an open neighborhood of x in X whose closure in υX is compact, so by the Hager-Johnson theorem its closure in X is a pseudocompact neighborhood of x in X.

The necessity is not much harder, although a complete proof involves checking a few details which have no independent interest. Let me outline it quickly for you. For each point x in X we select a pseudocompact neighborhood U_x of x and a continuous function f_x mapping X into $[0, 1]$ such that $f_x(x) = 0$ and $f_x \equiv 1$ off U_x. If g_x is the continuous extension of f_x with domain υX, let $V_{x,r}$ denote, for $0 < r < 1$, the set $g_x^{-1}([0, r))$. Finally, define

$$Y = \bigcup_{\substack{x \in X \\ 0 < r < 1}} \mathrm{cl}_{\upsilon X} V_{x,r} .$$

It's obvious that $X \subset Y \subset \upsilon X$. The local compactness of Y follows from the fact that if, for a fixed $r < 1$, the number s is chosen so that $r < s < 1$, then $\mathrm{cl}_{\upsilon X} V_{x,r} \subset \subset V_{x,s}$, and the closure of the latter set in Y, being both pseudocompact and realcompact, is actually compact. This shows that each point of Y admits a compact neighborhood in Y, so I'm sure we can agree: Y is locally compact.

There is an easy corollary to the theorem just proved. Here it is.

Corollary. *If υX is locally compact, then X is locally pseudocompact.*

At the end of this paper I want to describe for you a space which constitutes a counterexample to the converse of that corollary. As a matter of fact, the space I have in mind is locally compact, but its Hewitt realcompactification is not even locally pseudocompact. Before passing to that space, however, I'd like to mention another theorem whose proof, because it involves no ideas not presented above, I will omit. This theorem is not so elegant as it might at first appear to you to be. For, although it characterizes the local compactness of the space υX and is so far as I am aware the only such theorem known, it does this in terms of a property of υX, not of X. Anthony Hager, in his doctoral dissertation [5], gives a condition on X equivalent to the condition that υX be both locally compact and σ-compact.

Theorem. *In order that υX be locally compact, it is necessary and sufficient that for each point p in υX there exist pseudocompact subsets A and B of X for which* (i) *$p \in \mathrm{cl}_{\upsilon X} A$ and* (ii) *some function in $C(X)$ assumes the value* 0 *throughout A and the value* 1 *throughout $X \setminus B$.*

The example with which I will conclude this lecture was concocted to answer in the negative the following question: "If X is locally compact, must υX also be locally compact?" When I described this space in a letter to Hugh Gordon, his letter of acknowledgement referred to the space as a "spiral staircase". Now you and I probably think of a staircase as being a subset of Euclidean 3-space; but the space I am about to describe, since it contains copies of a certain ordinal space which is well-known not to be metrizable, is not. Of course Gordon intended his epithet as a conceptual aid not to be taken literally, and in this spirit I recommend it to you highly. You will notice, however, that though our staircase has infinitely many steps, it will be difficult to climb far upon it. For the steps are all of the same size, and there is one particular point which lies on every step.

Let **W** denote the set of ordinal numbers less than the first uncountable ordinal number ω_1, and let **W*** denote the Stone-Čech compactification of **W**. Let Y be the space obtained from the product space $N \times \mathbf{W}^* \times \mathbf{W}^*$ by identifying, for each positive integer k and each ordinal number $\gamma \leqq \omega_1$, the two points

$$(k, \omega_1, \gamma) \quad \text{and} \quad (k+1, \gamma, \omega_1)\,.$$

Let p be the "center point"

$$(1, \omega_1, \omega_1) = (2, \omega_1, \omega_1) = \ldots = (k, \omega_1, \omega_1) = \ldots$$

The topology on Y is defined by the decree that a subset of Y, whether or not it contains the point p, is open if and only if it meets each set of the form $\{k\} \times \mathbf{W}^* \times \times \mathbf{W}^*$ in a relatively open subset. This definition turns Y into a completely regular Hausdorff space which is σ-compact, hence Lindelöf and hence realcompact. The point p has no compact neighborhood in Y, nor indeed even a pseudocompact one, since any neighborhood of p contains for some nonlimit ordinal number γ the sequence $\{(k, \gamma, \gamma)\}_{k \in N}$ as an open-and-closed set of isolated points.

I promised to describe for you a space X which is locally compact, and instead we seem to have a space Y which is not even locally pseudocompact. You have probably already guessed correctly at the definition of X. It is to be the space $Y \setminus \{p\}$, which is surely locally compact. To show that $Y = \upsilon X$, we need only check that each function in $C(X)$ extends continuously to Y. Such a function f is continuous on each set of the form $\{k\} \times \mathbf{W}^* \times \mathbf{W}^* \setminus \{p\}$, and a familiar theorem from the topological theory of ordinal spaces assures us that for each positive integer k there exists $\gamma_k < \omega_1$ such that

$$f(k, \sigma, \tau) = f(k, \gamma_k, \gamma_k)$$

whenever $\gamma_k \leqq \sigma \leqq \omega_1$ and $\gamma_k \leqq \tau \leqq \omega_1$ and $(\sigma, \tau) \neq (\omega_1, \omega_1)$. Since $\sup_k \gamma_k < \omega_1$, there is a deleted neighborhood of p on which f is constant, so that f certainly extends continuously to p.

In an interesting construction which will appear in [7], Norman Noble exhibits a k-space X whose realcompactification υX is not a k-space. Unfortunately his space X is not locally compact, and unfortunately the space Y I constructed for you a moment ago is a k-space (as you can easily verify for yourself). Thus neither his example nor mine can be used to provide an answer to the following question, which so far as I am aware is unsolved in set theories free of measurable cardinals. (A discrete space of measurable cardinality, if such a space exists, yields a negative answer to this question.) If X is locally compact, must υX be a k-space?

Notes added November 10, 1966. 1. I learned yesterday from John Mack in Lexington, Kentucky, U.S.A., that he has known about the space X constructed above since early 1964, when an anonymous referee guided him to it. To Mack, the space is of interest because it is a "locally compact non weak cb space such that υX is weak cb." It is discussed from this point of view in his joint paper with D. G. Johnson, The Dedekind completion of $C(X)$, to appear in the Pacific Journal of Mathematics.

2. Another anonymous referee has provided me with a solution to the problem raised in the last sentence of this paper. There is, in fact, a locally compact space X of nonmeasurable cardinal for which υX is not a k-space. Details will appear in my forthcoming paper, On the Hewitt realcompactification of a product space, to appear in the Transactions of the American Mathematical Society.

Note added in proof, April 14, 1967. Recent conversations reveal that the technique introduced above, or one very similar to it, is, in fact, at least two decades old. Using an argument suggested by Richard Arens, Edwin Hewitt uses a pasting argument much like ours, on spaces which are products of spaces indexed by ordinal numbers, to construct a regular space on which each continuous real-valued function is constant. His paper appears in Annals of Math. *47* (1946), 503—509.

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