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STRUCTURE OF TCHEBYCHEV SETS[1])

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In 1935 T. S. Motzkin proved that a Tchebychev set in Euclidean n-space is necessarily convex. Since then interest has developed in determining the widest class of Banach spaces in which all Tchebychev sets are convex. It appears plausible, for example, that all uniformly convex, smooth Banach spaces belong to this class. In fact, however, it has not been established that any infinite dimensional space admits only convex Tchebychev sets. Results have been obtained for the similar problem of determining classes of Tchebychev sets which are always convex. For example Vlasov [10] has shown that a boundedly compact Tchebychev set in a smooth Banach space is convex. V. Klee [6] has shown that this problem is related to the problem of determining when Tchebychev sets admit continuous metric projections. It is elementary to verify that a boundedly compact Tchebychev set admits a continuous metric projection. However it is not known, even in Hilbert space, whether every Tchebychev set admits a continuous metric projection.

The results in this paper will appear in a later study. Consequently the theorems below are not presented in their most general form, nor are the proofs given in detail.

Let $(X, \|.\|)$ be a normed linear space. A subset T of X is a *Tchebychev set* if for each x in X there is precisely one t in T such that $\|t - x\| = d(T, x) = \inf_{y \text{ in } T} \|y - x\|$. The *metric projection* associated with a Tchebychev set T is the function

$$\{(x, t) : x \text{ in } X,\ t \text{ in } T, \text{ and } \|x - t\| = d(T, x)\}.$$

Efimov and Stechkin have defined a set T to be *approximatively compact* if every minimizing sequence in T (i.e. a sequence $\{t_i\}$ such that $\|x - t_i\| \to d(T, x)$ for some x in X) admits a convergent subsequence. It is always true that an approximatively compact Tchebychev set admits a continuous metric projection. We will show that a locally compact Tchebychev set with a continuous metric projection is approximatively compact. First we define a catalytic property for Tchebychev sets. We call a subset of X *boundedly connected* if it intersects every open sphere in a connected set. This concept is utilized in the proof of each theorem below.

[1]) This paper contains results found in the authors dissertation written while he held a National Science Foundation Co-operative Fellowship. The author is grateful to Professor E. W. Cheney who supervised this study.

Theorem. *The following are equivalent for a locally compact Tchebychev set T:*

(i) *T admits a continuous metric projection,*

(ii) *T is boundedly connected,*

(iii) *T is approximatively compact.*

Furthermore if $(X, \|.\|)$ is a uniformly smooth and uniformly convex Banach space, then each of the above is equivalent to the convexity of T.

Proof. We will show that (ii) implies (iii).

Let x be in X. Let t be the point in T such that $\|t - x\| = \mathrm{d}(T, x)$. Let U be a neighborhood of t with compact closure. Let $\mathrm{b}U$ denote the boundary of U. Since $\mathrm{b}U$ is compact there is an $r > \|t - x\|$ such that the open sphere $S(r, x)$ of radius r and centered at x, does not meet $\mathrm{b}U$. Since $S(r, x) \cap T$ is connected and contains t, $S(r, x) \subseteq U$. If s is such that $r > s > \|t - x\|$, then the closed sphere of radius s centered at x intersects U in a compact set. It follows that T is approximatively compact. The second statement of the theorem follows directly from a theorem due to Efimov and Stechkin [4]. Their result states the equivalence of approximative compactness and convexity for a Tchebychev set in a uniformly smooth and uniformly convex Banach space.

The following corollary is a direct corollary to the above theorem and to a theorem of V. Klee [6]. We replace Klee's hypothesis of a continuous metric projection with the hypothesis of a locally compact, boundedly connected Tchebychev set.

Corollary. *Let T be a locally compact, boundedly connected Tchebychev set in a smooth reflexive Banach space X. If each point in $X \sim T$ admits a neighborhood on which the metric projection is weakly continous, then T is convex.*[2])

Corollary. *A compact Tchebychev set is the continuous image of $[0, 1]$.*

Proof. A compact Tchebychev set is boundedly connected. Hence it is connected and locally connected. An application of the Hahn-Mazurkiewicz theorem completes the proof.

By standard arguments a convex set in a strictly convex, reflexive Banach space is a Tchebychev set. If in addition the Banach space is uniformly convex a convex set admits a continuous metric projection. For the same conclusion with fewer restrictions on the norm see Ky Fan and Glicksberg [5], theorem 8. Since a convex set is boundedly connected we have the following corollary.

Corollary. *In any normed linear space a convex, locally compact Tchebychev set admits a continuous metric projection.*

[2]) Added in Proof: Recently Professor P. D. Morris has proved the aforementioned result by Klee without assuming the existence of a continuous metric projection. It follows that the locally compact, boundedly connected hypotheses in this Corollary are not necessary.

Theorem. *A closed set T in a uniformly convex (resp. uniformly convex and smooth) Banach space $(X, \|.\|)$ is convex if and only if in every equivalent uniformly convex (resp. uniformly convex and smooth) norm topology on X, T is a Tchebychev set which admits a continuous metric projection.*

Proof. If T is not convex, one can construct a norm d on X such that: (1) d is uniformly convex and smooth with $\|.\|$, (2) d and $\|.\|$ generate equivalent topologies on X, and (3) T is not boundedly connected in (X, d). It follows that T can not be a Tchebychev set with a continuous associated metric projection in (X, d).

Theorem. *A boundedly compact set in a strictly convex (strictly convex and smooth) normed linear space X is convex if and only if it is a Tchebychev set in all equivalent strictly convex (strictly convex and smooth) norm topologies on X.*

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