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REMARKS ON PRODUCT SPACES

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There are many kinds of sufficient conditions on spaces X and Y for the normality of the product space $X \times Y$. There are also many necessary conditions for the normality, but many of them are of the following kind: conditions on X in order that $X \times Y$ be normal for all Y in a class of spaces, or conditions including requirements on $X \times Y$ itself; in other words, there are only few necessary conditions on X and Y separately for the normality of $X \times Y$.

Introducing a concept of "upper compactness", we shall, in this note, state some propositions which may suggest the reason why we have only few such necessary conditions. We also give some properties of upper compactness and sufficient conditions for the normality expressed by using this concept, some of which are generalizations of known results.

In the following, spaces X and Y are, unless otherwise specified, Hausdorff. Let $\mathfrak{N}_a$, $a \in X$, be a neighborhood system of a in X, let Z be a subset of X, and let $\{B_x; x \in Z\}$ be any family of subsets $B_x \subset Y$. Then we define (cf. [4, § 8,2])

$$\limsup_{\mathfrak{N}_a} B_x = \bigcap_{U \in \mathfrak{N}_a} \overline{\bigcup_{x \in U} B_x}\,.$$

Definition 1. A space Y with the following property is called to be *upper compact with respect to* X. If Z is any non empty subset of X, and if $\{B_x; x \in Z\}$ is any family of non empty $B_x \subset Y$, then

$$\limsup_{\mathfrak{N}_a} B_x \neq \emptyset$$

for any $a \in \bar{Z}$.

The property that Y is upper compact with respect to X has close relations with the upper semi-continuity of a mapping of X to $\exp Y$.

It is easily verified that any space is upper compact with respect to any discrete space. Furthermore, we have

Proposition 1. *A space is compact if and only if it is upper compact with respect to any space.*

Proposition 2. *If every point of a space X has a neighborhood basis of power $\leqq \mathfrak{m}$, and if Y is $\mathfrak{m}$-compact, then Y is upper compact with respect to X.*

Proposition 3. *Let X be paracompact, and let Y be normal and upper compact with respect to X. Then $X \times Y$ is normal.*

From this proposition and Proposition 2, we have the following corollary, which was proved by K. Morita [3], Th. 4.1.

Corollary (Morita). *Let X be a paracompact space such that each point has a neighborhood basis of power $\leqq \mathfrak{m}$, and let Y be an $\mathfrak{m}$-compact normal space. Then $X \times Y$ is normal.*

We can generalize Proposition 2; for it, we need

Definition 2. Let $\mathfrak{m}$ be any cardinal number. A point x of a space X is said to be an $\mathfrak{m}$-*point* provided that if A is any subset of X with $x \in \bar{A}$, and if $\mathscr{F} = \{U\}$ is any family of neighborhoods of x with power $\leqq \mathfrak{m}$, then

$$A \cap \left(\bigcap_{U \in \mathscr{F}} U \right) \neq \emptyset .$$

Every point is an $\mathfrak{m}$-point for a finite $\mathfrak{m}$, and the so-called P-point of Gillman and Jerison [1] is $\aleph_0$-point in our terminology.

Proposition 4. *Let Y be any $\mathfrak{c}$-compact space, $\mathfrak{c} \geqq 1$, and let $\bar{\bar{Y}} = \mathfrak{m}$. Then Y is upper compact with respect to any space X with the following property: Every point x of X has a neighborhood basis of power $\leqq \mathfrak{c}$ or x is an $\mathfrak{m}$-point.*

Any space Y is $\mathfrak{c}$-compact at least for a finite $\mathfrak{c}$, so we can, in general, construct a non trivial space X with the property stated in this proposition; namely, we have

Corollary 1. *Any space is upper compact with respect to some non discrete paracompact space.*

Applying this corollary to Proposition 3, we have

Corollary 2. *Let Y be any normal space, then $X \times Y$ is normal for some non discrete paracompact space X.*

This corollary may suggest the reason of the shortage of desired necessary conditions for the normality of $X \times Y$, and may be useful when we try to find such conditions.

As we have seen above, there are several cases where compactness can be replaced by upper compactness. Let us mention another one below which is a generalization of K. Morita's result ([2], Th. 4) that is stated with "locally compact" instead of "locally upper compact with respect to X".

Definition 3. A space Y is called *locally upper compact with respect to X* if every point of Y has a neighborhood U such that $\bar{U}$ is upper compact with respect to X.

Proposition 5. *Let X be paracompact, and let Y be paracompact and locally upper compact with respect to X. Then $X \times Y$ is normal.*

References

[1] *L. Gillman and M. Jerison:* Rings of continuous functions. D. Van Nostrand, New York 1960.
[2] *K. Morita:* On the dimension of product spaces. Amer. J. Math. *75* (1953), 205—223.
[3] *K. Morita:* Paracompactness and product spaces. Fundam. Math. *50* (1961), 223—236.
[4] *G. Nöbeling:* Grundlagen der analytischen Topologie. Springer, Berlin 1954.