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A NEW DIMENSION TYPE

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In 1910 M. Fréchet [3] introduced the concept of dimension type in analogy with the theory of cardinal numbers in order to compare topological spaces. The Fréchet dimension type of a topological space X is said to be less than or equal to the Fréchet dimension type of a topological space Y if and only if X can be topologically embedded in Y. In this case we write $\mathrm{d}X \leqq \mathrm{d}Y$. A shortcoming of Fréchet dimension type is that many spaces are not comparable due to the need to be able to embed one space in the other. I modified this condition in [11] to obtain a new dimension type called quasi dimension type. Many more spaces are comparable with respect to quasi dimension type and yet many properties of Fréchet dimension type are preserved.

For two topological spaces X and Y, we say X is *quasi embeddable in* Y, if, for each covering α of X, there is a closed α-map of X into Y. A continuous function $f : X \to Y$ is an α-map if there is a covering β of Y such that $f^{-1}[\beta]$ refines α. We use "covering" to mean "open covering". We say *the quasi dimension type of a topological space X is less than or equal to the quasi dimension type of a topological space Y* if and only if X is quasi embeddable in Y. In this case we write $\mathrm{q}X \leqq \mathrm{q}Y$. If $\mathrm{q}X \leqq \mathrm{q}Y$ and $\mathrm{q}Y \leqq \mathrm{q}X$, then $\mathrm{q}X = \mathrm{q}Y$. Note that quasi dimension type is a topological invariant and is monotone on closed subsets (i.e., if X a closed subset of Y, then $\mathrm{q}X \leqq \mathrm{q}Y$). Given a compact metric space X, a space Y and a map $f : X \to Y$ into Y and a number $\varepsilon > 0$, we say that f is an ε-mapping of X into Y provided the diameter $\operatorname{diam} f^{-1}(y) < \varepsilon$, for each y in Y. For such spaces $\mathrm{q}X \leqq \mathrm{q}Y$ if for each $\varepsilon > 0$ there is an ε-mapping of X into Y.

In 1926 C. Kuratowski [5] showed that there are $2^{\mathfrak{c}}$ Fréchet dimension types represented by subsets of the real line. In [11] I showed that there are only denumerably many quasi dimension types represented by subsets of the real line. Furthermore, I completely determined the partial ordering of these types and gave a topological characterization of the linear sets having a given type. In [12] a snake-like continuum is constructed with $2^{\mathfrak{c}}$ quasi dimension types represented by its subsets.

In 1930 C. Kuratowski [6] characterized 1-dimensional polyhedra which are embeddable in the plane R^2 as those which do not contain either of the two primitive skew curves K_1 or K_2. The polyhedron K_1 is the 1-skeleton of a tetrahedron with the midpoints of a pair of non-adjacent edges joined by a segment and K_2 is the complete graph on five vertices. He also described the secondary skew curves K_3 and K_4 which are non-polyhedral 1-dimensional Peano continua. In 1937 S. Claytor [1] showed that a Peano continuum which is not a 2-sphere is embeddable in R^2 if and only if it does not contain any one of the four skew curves K_i, $i = 1, 2, 3, 4$. Although Claytor's result applies to polyhedra, it makes use of K_3 and K_4 and it seemed desirable to have a polyhedral version which does not consider objects other than compact polyhedra. S. Mardešić and I did this in [7] by replacing K_3 and K_4 by a polyhedron $\perp$ called the "spiked disc." It consists of a disc and an arc which have only one point in common and this point is an interior point of the disc and an end-point of the arc. We then proved the following theorem.

Theorem. *For a polyhedron P the following three statements are equivalent:*

(a) $\mathrm{d}P \leqq \mathrm{d}R^2$.
(b) $\mathrm{q}P \leqq \mathrm{q}R^2$.
(c) *P does not contain $K_1, K_2, \perp$ or S^2.*

This is proved by showing (a) and (b) are equivalent to (c). This theorem is related to the following problem.

For each pair of positive integers (n, m), $n \leqq m$, consider the following statement: If an n-dimensional polyhedron P quasi embeds in R^m, then it embeds in R^m. For which pairs (n, m) is the statement true? The previous theorem shows that the statement is true in the case (2,2). T. Ganea [4] has shown that the statement is true for $(n, 2n)$, $n \neq 2$. However, Mardešić and I [8] modified an example of P. M. Rice [10] to obtain polyhedra which show that the statement is false for the case (n, n), $n \geqq 4$.

In order to exhibit such polyhedra P we use the fact that for every $n \geqq 4$ there exists a combinatorial n-manifold M with boundary ∂M having the following properties

(1) M is contractible,
(2) $\pi_1(\partial M) \neq 1$,
(3) $M \times I \approx I^{n+1}$

(see [9] and [2]). We define P as the cone $C(\partial M)$ over ∂M and show it quasi embeds but fails to embed in R^n (see [8]).

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