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THE SPACE βN AND THE CONTINUUM HYPOTHESIS

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The following results about the spaces $\mathbf{N}^* = \beta\mathbf{N} - \mathbf{N}$ and $\mathbf{R}^* = \beta\mathbf{R} - \mathbf{R}$ are among those established several years ago with the aid of CH (the continuum hypothesis):

1. The spaces $\mathbf{N}^*$ and $\mathbf{R}^*$ contain P-points (i.e., points at which every continuous real function is locally constant) as well as non-P-points; therefore, since homeomorphisms carry P-points to P-points, these spaces are not homogeneous [9]. More generally, if X is any non-pseudocompact space (i.e., X admits an unbounded continuous function), then $\beta X - X$ is not homogeneous [7]. (Remark: $\beta X - X$ need not have P-points.)
2. All residue-class fields of $C(\mathbf{N})$ and $C(\mathbf{R})$ associated with the points of $\mathbf{N}^*$ or $\mathbf{R}^*$ (more generally, all real-closed η_1-fields of cardinal $\aleph_1$) are isomorphic [2].
3. $\mathbf{R}^*$ contains "remote" points, i.e., points not in the closure of any discrete subset of $\mathbf{R}$; equivalently, there exist z-ultrafilters on $\mathbf{R}$ none of whose members is nowhere dense [4].
4. No proper dense subset of $\mathbf{N}^*$ or $\mathbf{R}^*$ is C^*-embedded [3]. An open set in $\mathbf{N}^*$ or $\mathbf{R}^*$ is C^*-embedded (if and) only if it is a cozero-set [3].

These results are related to one another and generate many interesting problems. A major class of problems is to find the relations between CH and the cited or new results:

1. Can one show without CH that $\mathbf{N}^*$ has P-points?

M. E. Rudin showed recently that the non-P-points of $\mathbf{N}^*$ fall into 2^c types, as follows: all homeomorphisms of $\mathbf{N}^*$ onto itself carry each point of $\mathbf{N}^*$ to a point of the same type (unpublished). The proof uses CH; Z. Frolík has since obtained this result without CH (to appear in Bull. Amer. Math. Soc. [10]).

Can Frolík's result be applied to establish the non-homogeneity of $\beta X - X$ for all non-pseudocompact X?

2. Assume that CH is false. Can the isomorphism of the residue-class fields of $C(\mathbf{N})$ and $C(\mathbf{R})$ be established? Can it be established within the class of fields corresponding to P-points? Corresponding to non-P-points? Are the fields in the one class non-isomorphic to those in the other? Can the fields be thus differentiated among the 2^c types of points?

N. J. Bloch, in his Rochester thesis (1965), proved with the help of CH that for any two residue-class fields of $C(\mathbf{N})$ or $C(\mathbf{R})$ corresponding to points of $\mathbf{N}^*$ or $\mathbf{R}^*$, there is an isomorphism of one onto the other that carries a prescribed copy of $\mathbf{R}$ (among the infinitely many) in the first field to a prescribed copy in the second. (More generally, there is an isomorphism of any real-closed η_1-field of cardinal $\aleph_1$ onto any other, such that a prescribed subfield in which all well-ordered subsets are countable is carried onto a prescribed isomorphic subfield.) What is the relation of this to CH?

In the other direction, Bloch showed that if CH is false, then there exist two real-closed η_1-fields of cardinal c that are not isomorphic (in fact, not even similar as ordered sets). We do not know, however, whether these fields arise as residue-class fields of some $C(X)$.

3. D. L. Plank, in his Rochester thesis (1966), has devised a general method of associating certain classes of points of $\mathbf{N}^*$ and $\mathbf{R}^*$ with certain subalgebras of $C(\mathbf{N})$ and $C(\mathbf{R})$. Remote points and P-points emerge as special cases, as do the "B-points" arising from the ring

$$\{f \in C(\mathbf{N}) : \limsup |f(n)|^{1/n} \leqq 1\}$$

considered by Brooks [1; 8]. The existence of all these points rests on CH. The relations among the classes are not fully known. (E.g., all B-points are P-points, but the reverse question has not been settled.) The general method of associating classes of points with subalgebras can be exploited further. What results can be obtained without CH?

4. Is CH needed for the quoted results about C^*-embedding? (Remark: It is not needed to prove that cozero-sets in $\mathbf{N}^*$ or $\mathbf{R}^*$ are C^*-embedded [5].) Problem: Characterize the C^*-embedded subsets of $\mathbf{N}^*$ and $\mathbf{R}^*$ (with or without CH).

For the sake of convenience, and in order to point out the natural way in which CH comes up, I now present the proof that dense sets in $\mathbf{N}^*$ are not C^*-embedded. Let p be an arbitrary point of $\mathbf{N}^*$; it is sufficient to show that $\mathbf{N}^* - \{p\}$ is not C^*-embedded in $\mathbf{N}^*$. In fact, we shall show that there is a continuous 2-valued function on $\mathbf{N}^* - \{p\}$ with no continuous extension to p. Equivalently: we decompose $\mathbf{N}^*$ into disjoint sets A, B, and $\{p\}$, where A and B are open sets whose closures both contain p.

We proceed by transfinite recursion. The c zero-set-neighborhoods of p form a base at p; if CH is true, they can be indexed $(Z_\alpha)_{\alpha<\omega_1}$. In the key step, we are given $\alpha < \omega_1$ and assume that cozero-sets A_σ and B_σ have been defined for all $\sigma < \alpha$ such that

$$(*) \qquad p \notin A_\sigma \cup B_\tau \quad \text{and} \quad A_\sigma \cap B_\tau = \emptyset$$

for all $\sigma, \tau < \alpha$. Now, in any space, countable unions of cozero-sets are cozero-sets; and in the space $\mathbf{N}^*$, disjoint cozero-sets can be separated by a partition. (See [6] for background.) Consequently, there exist complementary open-and-closed sets A_α^0 and B_α^0 satisfying

$$A_\alpha^0 \supset \bigcup_{\sigma<\alpha} A_\sigma \quad \text{and} \quad B_\alpha^0 \supset \bigcup_{\sigma<\alpha} B_\sigma .$$

Next, the set

$$Z_\alpha - \bigcup_{\sigma<\alpha}(A_\sigma \cup B_\sigma)$$

contains p and hence is a nonvoid zero-set in $\mathbf{N}^*$; as such, it has a nonvoid interior. Since no point of $\mathbf{N}^*$ is isolated, the reduced set

$$Z_\alpha - \bigcup_{\sigma<\alpha}(A_\sigma \cup B_\sigma) - \{p\}$$

also has a nonvoid interior and so contains disjoint nonvoid cozero-sets A'_α and B'_α.

We now define

$$A_\alpha = (A^0_\alpha - Z_\alpha) \cup A'_\alpha \quad \text{and} \quad B_\alpha = (B^0_\alpha - Z_\alpha) \cup B'_\alpha\,.$$

These are cozero-sets and (*) holds for all $\sigma, \tau < \alpha + 1$.

Finally, we define

$$A = \bigcup_{\alpha<\omega_1} A_\alpha \quad \text{and} \quad B = \bigcup_{\alpha<\omega_1} B_\alpha\,.$$

These are disjoint open sets, neither containing p. Their union is all of $\mathbf{N}^* - \{p\}$, since for any $q \neq p$ there is a neighborhood Z_α of p that excludes q, whence $q \in A_\alpha \cup \cup B_\alpha$. And $p \in \mathrm{Cl}\, A \cap \mathrm{Cl}\, B$ because the basic neighborhood Z_α of p contains A'_α and B'_α.

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Added in proof:

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