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PROJECTIVE COVERS IN CERTAIN CATEGORIES OF TOPOLOGICAL SPACES

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Introduction. Projective objects have been studied in a number of categories of topological spaces and continuous (or more restricted) mappings, e.g. the category **C** of all compact Hausdorff spaces and their continuous mappings [3], the category **H** of all Hausdorff spaces and their proper, i.e., continuous, closed, and compact mappings [2], as well as a few others. It was shown in [3] that the extremally disconnected members of **C** are precisely the projective objects of **C**, and the analogous assertion for **H** was proved in [2]. Similarly, it was established in [3] that every object of **C** is the image, in a particular way, of a projective object of **C** (projective resolution or cover), and in [2] the corresponding result was obtained for the category **R** of regular Hausdorff spaces and their proper mappings.

Although the properties with respect to projectivity of, say, the category **H** are thus similar to those of its sub-category **C**, this is not a situation of the general-versus-special-case type since characterization of the projective objects in a category does not, in general, supply analogous knowledge concerning the projective objects in its sub-categories. A general result in this area will have to be concerned with a suitably specified class of categories which covers the categories previously discussed. It is the aim of this note to present such a result[1]).

The conditions which will be considered for a category **K** are as follows:

I. *All objects of* **K** *are Hausdorff spaces, and all mappings of* **K** *are proper.*

II. *If* $X \in \mathbf{K}$ *and* $f : X \to Y$ *is a homeomorphism then* $Y \in \mathbf{K}$ *and* $f \in \mathbf{K}$.

III. **K** *is closed under fibre products.*

IV. *If A and B are closed subspaces of* $X \in \mathbf{K}$ *then their topological sum* $A \oplus B$ *and the mapping* $i_A \oplus i_B$: $A \oplus B \to X$ *belong to* **K**, *where* i_A: $A \to X$ *and* i_B: $B \to X$ *are the natural injections.*

V. **K** *is closed under projective limits of inverse systems whose indexing sets are well-ordered and all whose mappings are onto.*

Examples of categories which satisfy these conditions will be given in Section 2.

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1. Projectivity. In the following, a mapping $f : X \to Y$ between spaces will be called:

a *projection* if $f(X) = Y$;
an *essential projection* if $f(A) \subset X$ for any closed $A \subset X$;[2])
a *projective cover* of Y (in a category **K**) if it is an essential projection such that there exist (in **K**) no non-trivial essential projections $f' : X' \to X$;
a *cross section* of a mapping $g : Y \to X$ if $g \circ f$ is the identity mapping on X.

Concerning essential projections one has three useful results which are employed in the proofs of the propositions of this section:

(i) The composite of essential projections is an essential projection.
(ii) If $f : X \to Y$ is a projection and each $f^{-1}(y)$, $y \in Y$, is compact then there exists a closed $X_0 \subseteq X$ such that $f \mid X_0 : X_0 \to Y$ is an essential projection.
(iii) If $f : X \to Y$ is an essential projection and X is Hausdorff then the cardinality of X is not greater than the number of all filters on Y.

In a category **K** for which I–IV hold one then obtains:

Proposition 1. *The following conditions are equivalent for an object $P \in$* **K**:

(1) *P is extremally disconnected.*
(2) *P is projective in* **K**.
(3) *Every projection $f : X \to P$ in* **K** *has a cross section in* **K**.
(4) *There exist no non-trivial essential projections $f : X \to P$ in* **K**.

Corollary 1. *An essential projection $f : X \to Y$ in* **K** *is a projective cover of Y in* **K** *iff X is projective in* **K**.

Corollary 2. *If $f : X \to Y$ is a projective cover in* **K** *and $g : Z \to Y$ any essential projection then there exists an essential projection $h : X \to Z$ such that $f = h \circ g$. Moreover, if g is also a projective cover then h is a homeomorphism.*

The last corollary establishes, in particular, the essential uniqueness of projective covers. Also, it states that projective covers, though defined in terms of minimality, are in fact least elements (with respect to a suitably given quasi-order).

In a category **K** which satisfies I–V, the question of the existence of projective objects, and, specifically, of projective covers is settled as follows:

Proposition 2. *Any object in* **K** *has a projective cover.*

The proof of this makes crucial use of V and of the remark, at the beginning of this section, regarding the cardinality of the domain of an essential projection.

These results have an interesting consequence for a sub-category **L** of a category **K** when both, **K** and **L**, satisfy I–V: $X \in$ **L** is projective in **L** iff it is projective in **K**, and $f : X \to Y$ in **L** is a projective cover in **L** iff it is a projective cover in **K**. Moreover, if $f : X \to Y$ is a projective cover in **K** and $Y \in$ **L** then f and X also belong to **L**.

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[2]) The symbol $\subset$ is used for proper subsets.

2. Specific categories. In this section, we indicate a number of categories which arise naturally in general topology and for which the conditions I – V hold. This is immediately obvious for the category **C**, and can be shown, with the aid of suitable results on proper maps from [1], for the category **H**, which thus turns out to be the most inclusive category satisfying I – V.

Concerning other categories one has, to begin with, the following:

Lemma 1. *Any full sub-category* **S** *of* **H** *whose class of objects has members with more than one point and is closed with respect to*

(1) *homeomorphic images,*
(2) *closed subspaces, and*
(3) *Cartesian products*

satisfies I – V.

Incidentally, the exclusion of categories containing only spaces with at most one point is necessary: such categories may fulfill the other hypotheses of this lemma although they evidently cannot satisfy IV.

From here, one now obtains that the conditions I – V hold for the full sub-categories of **H** determined by the following types of spaces:

(1) *regular spaces,*
(2) *completely regular spaces,*
(3) *zero-dimensional spaces,*
(4) *compact zero-dimensional spaces,*
(5) *real compact spaces.*

Another method of determining categories which satisfy I – V is based on a variant of the above lemma in which (3) is replaced by the condition: *If* $f : X \to Y$ *in* **H** *and* $Y \in$ **S** *then also* $X \in$ **S**. With this, the above list of types of spaces can be extended as follows:

(6) *locally compact Hausdorff spaces,*
(7) *locally compact paracompact Hausdorff spaces,*
(8) *σ-compact spaces.*

Examples of full sub-categories of **H** which do not satisfy I – V are those given by the metrizable spaces and the semi-regular spaces.

3. Projective covers as filter spaces. In [2] and [3], the existence of projective covers in the categories considered there was obtained by explicit descriptions of suitable spaces and mappings which were proved to provide the desired covers. In either case, the spaces were made up of filters in certain topologically defined lattices, i.e., the maximal filters in the Boolean lattices of all regular closed, or regular open, subsets of the initial space. In analogy with this, we now give a similar description of projective covers, applicable to any category which satisfies I – V; we use the same approach as in [4].

In the following, for a space X, let $\mathfrak{O}(X)$ be its topology, i.e., the collection of its open sets, and $\Omega(X)$ the space of maximal filters $\mathfrak{M} \subseteq \mathfrak{O}(X)$ whose topology is generated by the sets $\Omega_V(X) = \{\mathfrak{M} \mid V \in \mathfrak{M} \in \Omega(X)\}$ for $V \in \mathfrak{O}(X)$. $\Omega(X)$ is an extremally disconnected compact Hausdorff space. In $\Omega(X)$, consider the subspace $\Lambda(X)$ of all convergent $\mathfrak{M} \in \Omega(X)$, and denote by lim_X the mapping $\Lambda(X) \to X$ which assigns to each $\mathfrak{M} \in \Lambda(X)$ its limit. Then, $\Lambda(X)$ is extremally disconnected, and lim_X is a projection. Moreover, lim_X is compact, closed, essential, and for any $V \in \mathfrak{O}(X)$, the image of $\Lambda_V(X) = \Lambda(X) \cap \Omega_V(X)$ is the closure of V. In particular, lim_X is continuous iff X is regular.

The following two results constitute the most significant steps towards the desired description of projective covers.

Lemma 2. *For any proper essential projection $f : X \to Y$, the mapping $f^* : \Lambda(Y) \to \Lambda(X)$ which assigns to each $\mathfrak{M} \in \Lambda(Y)$ the filter generated by $f^{-1}(\mathfrak{M})$ in $\mathfrak{O}(X)$ is a homeomorphism.*

Lemma 3. *If X is an extremally disconnected Hausdorff space and X_* the space obtained from X by generating $\mathfrak{O}(X_*)$ with the regular $V \in \mathfrak{O}(X)$ then the mapping $\Lambda(X) \to X_*$ given by lim_X is a homeomorphism.*

From these lemmas one obtains, for a category **K** in which I—V hold:

Proposition 3. *If all $X \in$ **K** are semi-regular then, for each $X \in$ **K**, $\Lambda(X)$ and lim_X belong to **K** and $lim_X : \Lambda(X) \to X$ is a projective cover of X in **K**. In general, a projective cover for $X \in$ **K** is given by lim_X on the space $\Lambda'(X)$ whose underlying set is the same as that of $\Lambda(X)$ and whose topology is generated by that of $\Lambda(X)$ and $lim_X^{-1}(\mathfrak{O}(X))$.*

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