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CONVERGENCE STRUCTURES

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In recent years, a number of structures similar to but richer than the classical continuity structures (i.e. topology, proximity, uniformity) have drawn the attention of mathematicians. Only one of these has been treated monographically, namely the syntopogenic spaces; but a series of papers has appeared on various types of such structures.

This paper treats two types of continuity structures, namely the convergence structures (Definition 3) and the merotopies; the latter, considered by the author in [4], have already occurred implicitly in the papers [5] of K. Morita. In the present note, it is stated that convergence structures are equivalent, in a specified sense, to a type of merotopies. Therefore, an explicit mention of convergence structures is sometimes omitted, and the formulations concern rather the merotopies.

All proofs are omitted, and the references are strongly restricted. However, the papers of H. R. Fischer [3], C. H. Cook and H. R. Fischer [2] and H. Poppe [6] should be mentioned. Some notions and propositions contained in the present note are closely related to the content of these papers. As for further bibliography, we refer to [4] and [6].

Besides current terminology and notation, we use a few terms and symbols from [1]; these are introduced explicitly unless they are self-explanatory or well known. As usual we sometimes denote, e.g., a space and the set of its points by the same symbol.

1.

Definition 1. Let $M = \{x_a \mid a \in A\}$ be a net (i.e., a family indexed by a directed set). A net $N = \{y_b \mid b \in B\}$ is called a *quasi-subnet* of M if for any $a_0 \in A$ there is a $b_0 \in B$ such that $b \geqq b_0$ implies $y_b = x_a$ for some $a \geqq a_0$.

Remark. Clearly, every subnet or generalized subnet (e.g., in the sense of [1], p. 266) is a quasi-subnet. On the other hand, a quasi-subnet N of a net M need not be a generalized subnet of M. An example: let D be an uncountable set, let A consist of all finite sets $a \subset D$ and let A be ordered by inclusion; for any $a \in A$ put $x_a = \text{card}\, a$. Put $M = \{x_a \mid a \in A\}$; put $N = \{n \mid n \in \mathbb{N}\}$.

Definition 2. A non-void class A of non-void directed sets is called *admissible* iff it contains all subsets cofinal to, and also all isomorphs of its members.

Definition 3. Let A be an admissible class of directed sets. Let E be a set.

Let $\mathscr{L}$ be a class of pairs $\langle M, x \rangle$ such that

(0) if $\langle M, x \rangle \in \mathscr{L}$, then $x \in E$ and M is a net of elements of E;

(1) if $\langle M, x \rangle \in \mathscr{L}$, then M is of the form $M = \{x_a \mid a \in A\}$ for some $A \in A$;

(2) if $\langle M, x \rangle \in \mathscr{L}$ and $N = \{y_b \mid b \in B\}$, $B \in A$, is a quasi-subnet of M, then $\langle N, x \rangle \in \mathscr{L}$;

(3) if $x \in E$, $A \in A$ and $x_a = x$ for every $a \in A$, then $\langle \{x_a \mid a \in A\}, x \rangle \in \mathscr{L}$.

(4) if $\langle \{x_a \mid a \in A\}, x \rangle \in \mathscr{L}$, $y \in E$, $A' \subset A$ is cofinal and $x_a = y$ for $a \in A'$, then $\langle \{x_a \mid a \in A\}, y \rangle \in \mathscr{L}$; if $A \in A$, $A' \subset A$, $x \in E$, $x_a = x$ for $a \in A - A'$ and $\langle \{x_a \mid a \in A'\}, x \rangle \in \mathscr{L}$, then $\langle \{x_a \mid a \in A\}, x \rangle \in \mathscr{L}$.

Then $\mathscr{L}$ is called an *A-convergence structure* (or *A-convergence*) on E, and the pair $\langle E, \mathscr{L} \rangle$ is called an *A-convergence space.*

Remarks. 1) The concept just defined differs essentially from that introduced under the same name in [1], p. 645. Other related concepts based on the convergence of sets may be introduced, and each of them may possess certain specific advantages. The notion introduced in this note seems to be convenient if a close relationship with filter convergence is desirable.

2) Condition (4) is somewhat clumsy; unfortunately, if it is dropped, then the equivalence of convergence and LF-merotopic spaces asserted in Theorem 1 does not hold.

Definition 4. Let A_1, A_2, $A_1 \subset A_2$, be admissible classes of directed sets. Let $\mathscr{L}_i$ be an A_i-convergence on E_i, $i = 1, 2$. Then a mapping $f : \langle E_1, \mathscr{L}_1 \rangle \to \langle E_2, \mathscr{L}_2 \rangle$ is called *continuous* iff $\langle \{x_a \mid a \in A\}, x \rangle \in \mathscr{L}_1$ always implies $\langle \{fx_a \mid a \in A\}, fx \rangle \in \mathscr{L}_2$.

In Definition 3 and 4 there figures an external concept, namely that of an index set and of a class of such sets. To transpose the formulation so as to concern internal concepts only (i.e., formulated in terms of the basic set) is an interesting problem. A solution has been given by H. R. Fischer [3]. In this note a different approach is adopted, namely via the merotopic spaces [4].

We recall some basic definitions.

Definition (see [4], 1.4). Let E be a set. Let $\Gamma \subset \exp \exp E$ be such that (1) if $\mathscr{M} \in \Gamma$, $\mathscr{M}_1 \subset \exp E$ and to each $M \in \mathscr{M}$ there is an $M_1 \in \mathscr{M}_1$ with $M_1 \subset M$, then also $\mathscr{M}_1 \in \Gamma$; (2) if $\mathscr{M}_1 \cup \mathscr{M}_2 \in \Gamma$ then $\mathscr{M}_1 \in \Gamma$ or $\mathscr{M}_2 \in \Gamma$; (3) $((x)) \in \Gamma$ for all $x \in E$; (4) $(\emptyset) \in \Gamma$, $\emptyset \notin \Gamma$. Then Γ is called a *merotopic structure*, or *merotopy*, on E; $\langle E, \Gamma \rangle$ is termed a *merotopic space*. Members of Γ are said to be *micromeric*.

Definition (see [4], 1.7). If $\langle E_i, \Gamma_i \rangle$, $i = 1, 2$, are merotopic spaces, then a mapping $f : \langle E_1, \Gamma_1 \rangle \to \langle E_2, \Gamma_2 \rangle$ is called *continuous* if $f[\mathscr{M}] \in \Gamma_2$ whenever $\mathscr{M} \in \Gamma_1$.

Example 1. Let $\langle E, u \rangle$ be a topological space (or a closure space, see e.g. [1]). If $E \neq \emptyset$, let Γ_u consist of all $\mathcal{M} \subset \exp E$ such that, for some $x \in E$, every neighborhood of x contains some $M \in \mathcal{M}$; if $E = \emptyset$, let $\Gamma_u = ((\emptyset))$. Then Γ_u is a merotopy; we shall say that it is *induced* by the topology (or closure structure) u.

Example 2. Let $\langle E, \mathcal{U} \rangle$ be a uniform space ($\mathcal{U}$ is the collection of all uniform neighborhoods of the diagonal). Let Γ_u consist of all $\mathcal{M} \subset \exp E$ such that, for any $U \in \mathcal{U}$, there is an $M \in \mathcal{M}$ with $M \times M \subset U$. Then $\Gamma_{\mathcal{U}}$ is a merotopy; we shall say that $\Gamma_{\mathcal{U}}$ is *induced* by the uniformity $\mathcal{U}$.

Example 3. Let D be a set. Let $\mathcal{T}$ be a non-void collection of closed subsets of βD, and let (x) belong to $\mathcal{T}$ for any $x \in D$. Let $\Gamma(\mathcal{T})$ consist of all $\mathcal{M} \subset \exp D$ such that, for some $T \in \mathcal{T}$, every neighborhood of T in βD contains some $M \in \mathcal{M}$. Then $\Gamma(\mathcal{T})$ is a merotopy on D. – It can be shown that (i) $\Gamma(\mathcal{T})$ is a filter-merotopy (see below) (ii) if Γ is a filter-merotopy on D, then there exists exactly one $\mathcal{T}$ with the properties described above and such that $\Gamma = \Gamma(\mathcal{T})$.

Definition (see [4], 1.21). Let Γ be a merotopy on E; for every $X \subset E$ let $\tau_\Gamma X$ consist of all $x \in E$ such that, for some micromeric $\mathcal{M}$, $M \in \mathcal{M}$ implies $x \in M$, and $M \cap X \neq \emptyset$. Then τ_Γ is a closure structure on E (i.e., $\tau_\Gamma \emptyset = \emptyset$, $\tau_\Gamma(X \cup Y) = \tau_\Gamma X \cup \cup \tau_\Gamma Y$, $X \subset \tau_\Gamma X$); we shall say that τ_Γ is *induced* by the merotopy Γ.

Definition (see [4], 1.17). Let Γ be a merotopy on E. A system Θ, $\Theta \subset \Gamma$, will be called *fundamental* (for Γ) if $\Gamma \subset \Delta$ whenever Δ is a merotopy on E with $\Theta \subset \Delta$. A collection $\mathcal{B}$, $\mathcal{B} \subset \exp E$ will be called a *base* (for Γ) if there is a fundamental system Θ such that $\bigcup \Theta \subset \mathcal{B}$.

Remark. If Θ is a system of filters, then there exists exactly one merotopy for which Θ is fundamental.

Definition (see [4], 2.1). Let Γ be a merotopy on a set E. Then Γ is called a *filter-merotopy* and $\langle E, \Gamma \rangle$ is called a *filter-merotopic space* if there exists a fundamental system consisting of filters.

Definition 5. Let $\langle E, \Gamma \rangle$ be a merotopic space. We shall say that a micromeric collection $\mathcal{M}$ is *localized at a point* $a \in E$ iff the collection of all $M \cup (a)$, $M \in \mathcal{M}$ is micromeric. The merotopy Γ (and also the space $\langle E, \Gamma \rangle$) will be called *localized* iff either $E = \emptyset$ or every micromeric $\mathcal{M}$ is localized (at some point of E). A localized filter-merotopy (filter-merotopic space) will be also called an LF-merotopy (an LF-space).

Proposition 1. *Let A be an admissible class of directed sets. Let $\mathcal{L}$ be an A-convergence structure on a set E. If $\langle M, x \rangle \in \mathcal{L}$, $M = \{x_a \mid a \in A\}$, let $\mathcal{F}\langle M, x \rangle$ consist of all sets $X_a \cup (x)$, where X_a is the set of all $x_{a'}$ with $a' \in A$, $a' \geqq a$. Then the system of all collections $\mathcal{F}\langle M, x \rangle$, $\langle M, x \rangle \in \mathcal{L}$, is fundamental for a merotopy Γ and Γ is a localized filter merotopy.*

Definition 6. The merotopy Γ just described will be denoted by $\boldsymbol{\mu}\mathscr{L}$, and $\langle E, \boldsymbol{\mu}\mathscr{L}\rangle$ will be denoted by $\boldsymbol{\mu}\langle E, \mathscr{L}\rangle$.

Proposition 2. *Let A be an admissible class of directed sets. Let Γ be a merotopy on a set E. Let $\mathscr{L}$ consist of all pairs $\langle M, x\rangle$ such that* (i) $x \in E$, *M is a net* $\{x_a \mid a \in A\}$, $x_a \in E$, $A \in A$, (ii) *the collection of all sets $X_a \cup (x)$, $a \in A$, where X_a consists of all $x_{a'}$, $a' \in A$, $a' \geqq a$, is micromeric. Then $\mathscr{L}$ is an A-convergence structure on E.*

Definition 7. The convergence structure $\mathscr{L}$ just described will be denoted by $\lambda_A\Gamma$, and $\langle E, \lambda_A\Gamma\rangle$ will be denoted by $\lambda_A\langle E, \Gamma\rangle$.

Remark. Clearly, $\lambda_A\Gamma \neq \lambda_{A'}\Gamma$ if A, A' are two distinct admissible classes. It is also easy to see that, in general, even for LF-merotopies Γ, $\boldsymbol{\mu}\lambda_A\Gamma \neq \Gamma$.

Definition 8. Let A be an admissible class of directed sets. A merotopy Γ on a set E is said to be an *A-convergence-merotopy* if $\Gamma = \boldsymbol{\mu}\mathscr{L}$ for some A-convergence structure $\mathscr{L}$.

Theorem 1. *Let A be an admissible class of directed sets. If $\mathscr{L}$ is an A-convergence structure on a set E, then $\lambda_A\boldsymbol{\mu}\mathscr{L} = \mathscr{L}$. If Γ is an A-convergence merotopy on a set E, then $\boldsymbol{\mu}\lambda_A\Gamma = \Gamma$. If $\mathscr{L}_i$ is an A-convergence on E_i, $i = 1, 2$, and $f : \langle E_1, \mathscr{L}_1\rangle \to \langle E_2, \mathscr{L}_2\rangle$ is continuous, then $f : \langle E_1, \boldsymbol{\mu}\mathscr{L}_1\rangle \to \langle E_2, \boldsymbol{\mu}\mathscr{L}_2\rangle$ is also continuous. If Γ_i is a merotopy on E_i, $i = 1, 2$, and $f : \langle E_1, \Gamma_1\rangle \to \langle E_2, \Gamma_2\rangle$ is continuous, then $f : \langle E_1, \lambda_A\Gamma_1\rangle \to \langle E_2, \lambda_A\Gamma_2\rangle$ is continuous.*

Remark. The theorem (which is proved quite easily) asserts essentially that, for any admissible A, "the category of all A-convergence spaces" and that of all A-convergence-merotopic spaces are isomorphic. However, in the frame of the current theory of classes and sets, this statement is not correct since A-convergence structures are proper classes.

We are now going to consider the cartesian products.

Definition (see [4], 3.5). Let $Z \neq \emptyset$ be a set; let $\{E_z \mid z \in Z\}$ be a collection of non-void sets; we denote by pr_z the projection of the cartesian product $E = \prod\{E_z\}$ onto E_z. For any $z \in Z$ let Γ_z be a filter-merotopy on E. Let Θ be the system of all filters $\mathscr{F}$ on E such that, for any $z \in Z$, $\mathrm{pr}_z[\mathscr{F}]$ is micromeric (with respect to Γ_z). Let Γ denote the merotopy for which Θ is fundamental; then Γ is a filter merotopy. We shall call $\langle E, \Gamma\rangle$ the *cartesian product* of $\{\langle E_z, \Gamma_z\rangle\}$ and denote it by $\prod\{\langle E_z, \Gamma_z\rangle\}$

Now let an admissible class A be given. For every $z \in Z$ let $\mathscr{L}_z$ be an A-convergence structure on E_z. Let $\mathscr{L}$ be the class of all pairs $\langle M, x\rangle$ such that $x \in E$, $M = \{x_a \mid a \in A\}$ where $x_a \in E$ and for any $z \in Z$ the pair $\langle\{\mathrm{pr}_z\, x_a \mid a \in A\}, \mathrm{pr}_z\, x\rangle$ belongs to $\mathscr{L}_z$. Then $\mathscr{L}$ is an A-convergence structure on E. We shall call $\langle E, \mathscr{L}\rangle$ the *cartesian product* of $\{\langle E_z, \mathscr{L}_z\rangle\}$ and denote it by $\prod\{\langle E_z, \mathscr{L}_z\rangle\}$.

The following proposition is rather important, even though it is almost evident.

Theorem 2. *Let A be an admissible class of directed sets. Let $\{E_c \mid c \in C\}$ be a non-void family of non-void sets. For any $c \in C$, let Γ_c be a filter-merotopy on E_c and let $\mathscr{L}_c = \lambda A \Gamma_c$. Then $\prod_c\{\langle E_c, \mathscr{L}_c\rangle\} = \lambda_A \prod_c\{\langle E_c, \Gamma_c\rangle\}$.*

Remarks. 1) The equality $\prod\{\mu\langle E_z, \mathscr{L}_z\rangle\} = \mu \prod\{\langle E_z, \mathscr{L}_z\rangle\}$ does not hold in general (not even for the case of two factors). A well known example: A consists of N and all its isomorphs, $\langle E, \varrho\rangle$ is a metric space, $\mathscr{L}$ consists of all $\langle\{x_n \mid n \in N\}, x\rangle$ with $N \in A$, $x_n \in E$, $x \in E$, $\varrho(x_n, x) \to 0$. If $x \in E$ is non-isolated, $x_n \in E$, $x_n \neq x$ for every $n \in N$, and $x_n \to x$, then let $\mathscr{M}$ be the collection of all sets $X_{i,j}$, where $X_{i,j}$ consists of all points $\langle x_m, x_n\rangle$ with $m \geqq i$, $n \geqq j$. Clearly $\mathscr{M}$ is micromeric in $\langle E, \mu\mathscr{L}\rangle \times \langle E, \mu\mathscr{L}\rangle$, but is not such in $\mu(\langle E, \mathscr{L}\rangle \times \langle E, \mathscr{L}\rangle)$.

2) Observe that if Γ_c are A-convergence-merotopies, then $\mu \prod\{\langle E_c, \lambda_A \Gamma_c\rangle\}$ is the product of $\{\langle E_c, \Gamma_c\rangle\}$ in the category of all A-convergence-merotopic spaces.

3) Theorem 2 may be conceived as a special case of similar theorems concerning spaces of mappings of convergence space. However, we shall not consider these questions now.

2.

In what follows, convergence structures will seldom be mentioned explicitly. However, the results concerning localized filter merotopic spaces can be transposed, according to Theorem 1, to propositions on convergence spaces, although the formulations involving merotopies are sometimes more convenient.

The topics considered include embedding of LF-spaces, inductive generation, fine merotopies, certain separation properties. First we recall or introduce some definitions.

Definition 9. If $\langle E, \Gamma\rangle$ is a merotopic space, then a collection $\mathscr{V}$ consisting of subsets of E is called a *Γ-cover* or a *merotopic cover*, or merely a *cover*, iff to any $\mathscr{M} \in \Gamma$ there exist $M \in \mathscr{M}$ and $V \in \mathscr{V}$ with $M \subset V$. A system Θ of covers is termed *determining* (or *complete*) iff every cover $\mathscr{U}$ can be refined to some cover $\mathscr{V} \in \Theta$.

Definition 10. A merotopic space $\langle E, \Gamma\rangle$ is called *semi-separated* if $A \subset E$, $(A) \in \Gamma$, imply that A contains one point at most.

Definition 11. The least cardinality of bases of a merotopic space is called the *weight* of the space.

Definition 12. Let $\langle E, \Gamma\rangle$ be a merotopic space. A subset $X \subset E$ is called *functionally closed* iff to each $x \in E - X$ there is a continuous $f : E \to \mathrm{R}$ with $f[X] = (0)$, $fx = 1$. The space $\langle E, \Gamma\rangle$ is termed *completely regular* (in the weak sense — however this qualifier will be omitted since strong complete regularity will not occur here) iff the functionally closed sets constitute a base.

An A-convergence space $\langle E, \mathscr{L} \rangle$ is called *A-completely regular* iff $\mathscr{L} = \lambda_A \Gamma$ for some completely regular LF-space $\langle E, \Gamma \rangle$. The least weight of such a space $\langle E, \Gamma \rangle$ is termed the *A-weight* of the A-convergence space $\langle E, \mathscr{L} \rangle$. Analogously (using $\mu\lambda_A$ instead of λ_A) one defines the *A-completely regular* LF-*spaces*, and their *A-weights.*

Theorem 3. *Let* $\mathfrak{m}$ *be an infinite cardinal. Then there exists a completely regular semi-separated* LF-*space* $Y_{\mathfrak{m}}$ *with weight* $\mathfrak{m}$ *into which one can embed every completely regular semi-separated* LF-*space of weight* $\leqq \mathfrak{m}$.

Furthermore, if A is an admissible class of directed sets, then every A-completely regular semi-separated A-convergence space with A-weight $\leqq \mathfrak{m}$ *can be embedded into the A-convergence space* $\lambda_A Y_{\mathfrak{m}}$.

The spaces $Y_{\mathfrak{m}}$ will now be described; of the proof of this theorem only the basic ideas will be indicated.

One starts with the topological space $[\![0, 1]\!]^{\mathfrak{m}}$, denoted by K; let $0 \in K$ be the element with zero coordinates. Now take any set B of cardinality $\mathfrak{m}$, form K^B, and for every $\beta \in B$ take the set K_β of all $x \in K^B$ with $x(\beta) = 0$. Set $S = \bigcup K_\beta$ (thus S consists of all elements of K^B with 0 among their coordinates). On S one defines a merotopy thus: for each β choose some fundamental system Θ_β of the space $\mathscr{K}_\beta$ (considered as a space whose merotopy is induced by a topology), and then take $\Theta = \bigcup \Theta_\beta$ as a fundamental system for the space S. Obviously L has weight $\mathfrak{m}$, and it is easily verified that S is completely regular. Finally let $Y_{\mathfrak{m}}$ be the cartesian product $S^{\mathfrak{m}}$. That $Y_{\mathfrak{m}}$ is a completely regular LF-space follows from simple general theorems on products of merotopic spaces, which will be omitted here. The possibility of embedding any completely regular LF-space $\langle E, \Gamma \rangle$ of weight $\leqq \mathfrak{m}$ into $Y_{\mathfrak{m}}$ is based on the fact that such a space must have a complete system (with cardinality $\leqq \mathfrak{m}$) of covers $\mathscr{V}_\alpha$; each of which consists of $\leqq \mathfrak{m}$ functionally closed sets. These latter may then be written as $\varphi^{-1}[0]$ for appropriate continuous $\varphi : \langle E, \Gamma \rangle \to K$; the entire cover can then be described in terms of a map f into S (each member of the cover has the form $f^{-1}[K_\beta]$).

Definition (see [4], 1.13). Let A be a set; for every $a \in A$, let $\langle E_a, \Gamma_\alpha \rangle$ be a merotopic space. Let X be a set and for every $a \in A$ let $f_a : \langle E_a, \Gamma_a \rangle \to X$ be a mapping. Then there exists a finest merotopy Γ on X rendering continous all $f_a : \langle E_a, \Gamma_a \rangle \to \langle X, \Gamma \rangle$. The merotopy Γ is said to be *inductively generated* by the family $\{f_a\}$. If $\langle E, \Gamma \rangle$ is a merotopic space, X is a set, $f : \langle E, \Gamma \rangle \to X$ is a surjective mapping, and Δ is the merotopy generated by $\{f\}$, then $\langle X, \Delta \rangle$ is called the *quotient* of $\langle E, \Gamma \rangle$ relative to f; we denote $\langle X, \Delta \rangle$ by $\langle X, \Gamma \rangle / f$.

A surjective mapping $f : \langle E_1, \Gamma_1 \rangle \to \langle E_2, \Gamma_2 \rangle$ such that $\langle E_2, \Gamma_2 \rangle = \langle E_1, \Gamma_1 \rangle / f$ is called a *quotient mapping.*

Theorem 4. *Every merotopic space is the quotient of some uniform space* (i.e., of the merotopic space induced by a uniform space as described in Example 2).

Theorem 5. *Every localized filter-merotopic space $\langle E, \Gamma \rangle$ is the quotient of a topological space* (i.e., of the merotopic space induced by a topological space); *this latter may be taken as the sum of Hausdorff spaces with a unique non-isolated point*; *if $\langle E, \Gamma \rangle$ is semi-separated, then the quotient mapping may be taken so that the inverse images of points are closed sets.*

Proposition 3. *Let $\langle E, \Gamma \rangle$ be a semi-separated localized filter-merotopic space. The following two conditions are equivalent*:

(a) *$\langle E, \Gamma \rangle$ possesses an irreducible fundamental system, i.e. a fundamental system Θ such that no $\Theta' \subset \Theta$, $\Theta' \neq \Theta$, is fundamental,*

(b) *there exists a closure space $\langle X, u \rangle$ and a quotient mapping $f : \langle X, \Gamma_u \rangle \to \langle E, \Gamma \rangle$ such that no surjective partial mapping distinct from f is a quotient mapping.*

Remark. A space $\langle X, u \rangle$ with properties described in (b) may be called a "resolvent" of $\langle E, \Gamma \rangle$. In general, a space $\langle E, \Gamma \rangle$ may possess several distinct resolvents. The resolvent to be described seems to possess many convenient properties. Let Θ be an irreducible fundamental system. For any $\mathcal{M} \in \Theta$, $\bigcap \mathcal{M}$ is a singleton; we put $\bigcap \mathcal{M} = (f\mathcal{M})$. A closure structure u on Θ is determined as follows: a complete collection of neighborhoods of $\mathcal{M}_0 \in \Theta$ consists of all $H(\mathcal{M}_0, M)$ where $M \in \mathcal{M}_0$ and $H(\mathcal{M}_0, M)$ is the set of all $\mathcal{M} \in \Theta$ such that either (i) $f\mathcal{M} \in M$, $f\mathcal{M} \neq f\mathcal{M}_0$, or (ii) M belongs to $\mathcal{M}$. It is easy to see that $f : \langle \Theta, \Gamma_u \rangle \to \langle E, \Gamma \rangle$ possesses properties described in (b).

We now turn to fine merotopies (roughly speaking, a merotopy Γ on E is "fine" if there is no merotopy Γ' which is strictly finer than Γ and induces the same closure structure, and satisfies certain further conditions).

Definition 13. Let $\langle E_i, \Gamma_i \rangle$ be merotopic spaces, $i = 1, 2$. A mapping $f : \langle E_1, \Gamma_1 \rangle \to \langle E_2, \Gamma_2 \rangle$ is called (i) *quasi-compact* iff to any filter $\mathcal{M}$ in E with $f[\mathcal{M}] \in \Gamma_2$ there exists a Γ_1-micromeric filter $\mathcal{M}_1$ minorizing $\mathcal{M}$ (this means that for any $M \in \mathcal{M}$ there exists a $M_1 \in \mathcal{M}_1$ with $M_1 \subset M$), (ii) *G-quasi-compact*, if to any filter $\mathcal{M}$ of open sets in E with $f[\mathcal{M}] \in \Gamma_2$ there exists a Γ_1-micromeric filter $\mathcal{M}_1$ of open sets minorizing $\mathcal{M}$.

Remark. In the case of topologically induced merotopies a mapping is quasi-compact iff all inverse images of points are compact.

Definition (see [4], 2.4). A space $\langle E, \Gamma \rangle$ is called *regular* if for any $\mathcal{M} \in \Gamma$ the collection of all $\overline{M}$ for $M \in \mathcal{M}$ is micromeric.

Theorem 6. *Let $\langle E, u \rangle$ be a regular semi-separated topological space. Then there exists precisely one localized filter-merotopy Θ on E such that* (1) *Θ induces u,* (2) *the identity mapping $J : \langle E, \Theta \rangle \to \langle E, \Gamma_u \rangle$ is quasi-compact,* (3) *$\langle E, \Theta \rangle$ is regular,* (4) *Θ is the finest localized filter-merotopy with properties* (1) *to* (3).

A fundamental system for this merotopy Θ consists of all collections $\overline{\mathscr{G}}$ where $\mathscr{G}$ is any maximal filter of non-void sets in E, $\overline{\mathscr{G}}$ consists of all $\bar{G}$, $G \in \mathscr{G}$, and $\bigcap \overline{\mathscr{G}} \neq \emptyset$.

Definition 14. A merotopic space $\langle E, \Gamma \rangle$ is called *semi-regular* iff it has a base consisting of sets of the form $E - \tau_\Gamma(E - \tau_\Gamma X)$.

Theorem 7. *Let $\langle E, u \rangle$ be a semi-separated topological space. Then there exists precisely one localized filter-merotopy Θ^* on E such that (1) Θ^* induces u, (2) the identity mapping $J : \langle E, \Theta^* \rangle \to \langle E, \Gamma_u \rangle$ is G-quasi-compact, (3) $\langle E, \Gamma \rangle$ is semi-regular, (4) Θ^* is the finest localized filter-merotopy with properties (1) to (3).*

A fundamental system for Θ^* consists of all collections $\mathscr{G}^*$ where $\mathscr{G}$ is any maximal filter of non-void open sets in E such that $\bigcap \{\bar{G} \mid G \in \mathscr{G}\}$ is non-void (hence a singleton (x)) and $\mathscr{G}^*$ consists of all $(x) \cup G$, $G \in \mathscr{G}$.

Remarks. 1) If $\langle E. u \rangle$ is paracompact, then the following covers constitute a complete system of merotopic covers of $\langle E, \Theta \rangle : \mathscr{V} = \{\bar{V}_\alpha\}$ where V_α is a disjoint locally finite collection of open sets with $\bigcup \bar{V}_\alpha = E$.

2) By the preceding remark, $\langle E, \Theta \rangle$ is compact whenever $\langle E, u \rangle$ is compact.

3) It is not difficult to show that both Θ (the "fine regular localized merotopy" of $\langle E, u \rangle$) and Θ^* (the "fine semi-regular localized meorotopy" of $\langle E, u \rangle$) possess a resolvent. It may be shown that a suitable resolvent coincides, under certain fairly weak assumptions, with the "absolute" in the well-known sense of V. Ponomarev.

4) In the specification of the fundamental systems indicated in theorem 6 and 7, if one omits the condition $\bigcap \bar{G} \neq \emptyset$, then one obtains coarser merotopies, leading to familiar compactifications and H-closed extensions.

To conclude this section, some separation properties will be introduced.

Definition 15. Let $\langle E, \Gamma \rangle$ be a merotopic space. We shall consider the following conditions:

(H) If $\mathscr{M}_1 \in \Gamma$, $\mathscr{M}_2 \in \Gamma$ and $M_1 \cap M_2 \neq \emptyset$ whenever $M_i \in \mathscr{M}_i$, then $[\mathscr{M}_1] \cup \cup [\mathscr{M}_2]$ (i.e. the collection of all $M_1 \cup M_2$ with $M_i \in \mathscr{M}_i$) is micromeric;

(UH) For every Γ-cover $\mathscr{U}$ there is a Γ-cover $\mathscr{V}$ with the following property: if $V_1 \in \mathscr{V}$, $V_2 \in \mathscr{V}$, $V_1 \cap V_2 \neq \emptyset$, then $V_1 \cup V_2 \subset U$ for some $U \in \mathscr{U}$;

(SH) If Θ is a non-void system of finite non-void centred collections $\mathscr{T} \subset \exp E$, and every $\mathscr{M} \subset \exp E$ intersecting all collections $\mathscr{T} \in \Theta$ is micromeric, then the collection of all $\bigcup \mathscr{T}$, $\mathscr{T} \in \Theta$, belongs to Γ.

If condition (H) is satisfied, we shall call $\langle E, \Gamma \rangle$ a *Hausdorff* merotopic space. If condition (UH) or (SH) is satisfied, $\langle E, \Gamma \rangle$ will be called a UH-*merotopic* or SH-*merotopic* space, respectively.

Remarks. 1) If $\langle E, u \rangle$ is a Hausdorff closure space, then $\langle E, \Gamma_u \rangle$ is an SH-space. However, there exist completely regular Hausdorff LF-merotopic spaces which do not satisfy condition (SH).

2) A Hausdorff topological space need not be a UH-space.

3.

We shall now define the merotopic linear spaces. For convenience we shall consider real linear spaces only.

Definition 16. Let E be a linear space and let Γ be a merotopy on E such that

(1) if $\mathcal{M}_1$, $\mathcal{M}_2$ are micromeric, then the collection of all $[M_1] + [M_2]$ with $M_1 \in \mathcal{M}_1$, $M_2 \in \mathcal{M}_2$, is micromeric,

(2) if $\mathcal{M} \in \Gamma$ and $\mathcal{B}$ is localized micromeric in R, then the collection of all $[B] \, . \, [M]$ with $B \in \mathcal{B}$, $M \in \mathcal{M}$, is micromeric.

Then we shall say that Γ is an *admissible merotopy* for E and that $\langle E, \Gamma \rangle$ is a *merotopic linear space.*

Remarks. 1) If Γ is a filter-merotopy, then the above conditions express, of course, the continuity of the mappings $\langle x, y \rangle \to x + y$ (of $E \times E$ into E) and $\langle a, x \rangle \to a \, . \, x$ (of $\mathsf{R} \times E$ into E).

2) It is easy to see that every semi-separated merotopic linear space is a Hausdorff space.

Definition 17. Let E be a linear space. If Γ is an admissible merotopy on E, let Θ consist of all collections $\mathcal{K}$ obtained as follows: if $\mathcal{M} \in \Gamma$ and s is a mapping of $\mathcal{M}$ into E, then $\mathcal{K} = \mathcal{K}(\mathcal{M}, s)$ consists of all $s(M) + M$, $M \in \mathcal{M}$. There exists an admissible merotopy Γ^* for which Θ is a fundamental system. If every Γ^*-micromeric filter is localized, we shall say that $\langle E, \Gamma \rangle$ is L-*complete.*

Definition (see [4], 3.6). Given filter-merotopic spaces $\mathcal{X} = \langle X, \Gamma \rangle$ and $\mathcal{Y} = \langle Y, \Delta \rangle$, the set C of all continuous maps $f : \mathcal{X} \to \mathcal{Y}$ is endowed with the merotopy whose fundamental system consists of all filters $\mathcal{F}$ on C such that, for each $\mathcal{M} \in \Gamma$, the collection $\mathcal{F}[\mathcal{M}]$ (i.e., the collection of all $F[M]$ with $F \in \mathcal{F}$, $M \in \mathcal{M}$) belongs to Δ; the resulting filter-merotopic space is denoted by $\mathcal{Y}^{\mathcal{X}}$.

Remarks. 1) The merotopy just described is the coarsest of all merotopies on C rendering continuous the evaluation mapping $\{\langle f, x \rangle \to fx\}$.

2) It is easy to prove that if $\mathcal{X}$, $\mathcal{Y}$ are LF-spaces and $\mathcal{Y}$ is regular, then $\mathcal{Y}^{\mathcal{X}}$ is also a regular localized filter-merotopic space.

Definition 18. A subset X of a merotopic linear space $\langle E, \Gamma \rangle$ is called L-*functionally closed* iff, for every $x \in E$ with $x \notin X$, there exists a continuous linear function $f : \langle E, \Gamma \rangle \to \mathsf{R}$ such that $fx > \sup f[X]$. If L-functionally closed sets constitute a base, then $\langle E, \Gamma \rangle$ is called *basically convex* (or simply *convex*).

Theorem 8. *Let $\langle E, \Gamma \rangle$ be a filter-merotopic space. Then $\mathsf{R}^{\langle E, \Gamma \rangle}$ is an* L-*complete convex merotopic space satisfying condition* (SH).

Remarks. 1) For any $\varepsilon > 0$ and any $V \subset E$, let $\Phi(V, \varepsilon)$ denote the set of all $f \in \mathsf{R}^E$ such that $|fx| \leqq \varepsilon$ whenever $x \in V$. If $\varepsilon_1 > \varepsilon_2 > \ldots$, $\varepsilon_k \to 0$, and $\{\mathcal{V}_k\}$ is a sequence of covers of $\langle E, \Gamma \rangle$ such that $\mathscr{V}_{k+1}$ refines $\mathscr{V}_k$, let $\mathscr{F} = \mathscr{F}(\{\mathscr{V}_k\}, \{\varepsilon_k\})$ consist of all finite intersections of sets $\Phi(V, \varepsilon_k)$ with $V \in \mathscr{V}_k$. It is easy to see that every $\mathscr{F}$ is micromeric, and every filter in $\mathsf{R}^{\langle E, \Gamma \rangle}$ localized at 0 refines some $\mathscr{F}(\{\mathscr{V}_k\}, \{\varepsilon_k\})$.

2) If $\mathscr{M} \in \Gamma$, $\varepsilon > 0$, let $\mathscr{V}(\mathscr{M}, \varepsilon)$ consist of all $V(M, \varepsilon)$, $M \in \mathscr{M}$. It can be shown that, for every collection $\mathscr{S}$ micromeric in $\mathsf{R}^{\langle E, \Gamma \rangle}$ and localized at 0, every $\mathscr{M} \in \Gamma$, and every $\varepsilon > 0$, there exist $F \in \mathscr{F}$ and $M \in \mathscr{M}$ such that $F \subset V(M, \varepsilon) \in \mathscr{V}(\mathscr{M}, \varepsilon)$. In other words, every $\mathscr{V}(\mathscr{M}, \varepsilon)$ is a "merotopic cover at 0"; in addition, the collections $\mathscr{V}(\mathscr{M}, \varepsilon)$ form what may be called a "complete system of merotopic covers at 0".

Theorem 9. *Let a completely regular semi-separated localized filter-merotopic space $\langle E, \Gamma \rangle$ satisfy condition* (SH). *For an $x \in E$, let φx be the mapping of $\mathsf{R}^{\langle E, \Gamma \rangle}$ into R assigning fx to f. Then, with $L = \mathsf{R}^{\langle E, \Gamma \rangle}$, $\varphi : \langle E, \Gamma \rangle \to \mathsf{R}^L$ is an embedding, i.e. both φ and $\varphi^{-1} : \varphi[E] \to \langle E, \Gamma \rangle$ are continuous.*

Definition 19. If $L = \langle L, \Gamma \rangle$ is a filter-merotopic linear space, then L' denotes the linear space of all continuous linear forms on L endowed with the merotopy of a subspace of R^L.

Example. If L is a Banach space (more precisely, L is endowed with the merotopy induced by the topology of a Banach space), then L' is the space of continuous linear forms endowed with the merotopy induced by the following convergence: $\langle \{f_a \mid a \in A\}, f \rangle \in \mathscr{L}$ iff A is an arbitrary directed set and (i) $f_a x \to fx$ for each $x \in L$, (ii) there is an $a_0 \in A$ such that $\{f_a \mid a \in A, a \geqq a_0\}$ is bounded.

Theorem 10. *Let L be a filter-merotopic linear space. Then L' is an* L*-complete convex merotopic linear space satisfying condition* (SH). *Let φ denote the natural mapping of L into L'', i.e. the mapping which assigns to any $x \in L$ the form $F_x \in L''$ defined by $F_x f = fx$, $f \in L$. Then $\varphi[L]$ is dense in L'' and φ is continuous. The mapping φ is* (i) *injective if and only if every one-point subset of L is* L*-functionally closed,* (ii) *an embedding if and only if, in addition, L is convex and satisfies condition* (SH), (iii) *an isomorphism if and only if L is an* L*-complete semi-separated convex merotopic linear space satisfying condition* (SH).

References

[1] *E. Čech:* Topological Spaces, revised edition. Prague - London 1966.

[2] *C. H. Cook and H. R. Fischer:* On equicontinuity and continuous convergence. Math. Ann. *159* (1965), 95—104.

[3] *H. R. Fischer:* Limesräume. Math. Ann. *137* (1959), 269—303.

[4] *M. Katětov:* On continuity structures and spaces of mappings. Comm. Math. Univ. Carol. *6* (1965), 257—278.

[5] *K. Morita:* On the simple extension of a space with respect to a uniformity. I.—IV. Proc. Japan Acad. *27* (1951), 65—72, 130—137, 166—171, 632—636.

[6] *H. Poppe:* Stetige Konvergenz und Satz von Ascoli und Arzelà. Math. Nachr. *30* (1965), 87—122.