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CATEGORIAL METHODS IN TOPOLOGY

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Many "continuity structures" occur in mathematics in addition to the topological spaces (e.g. convergence spaces, proximity spaces, topogene spaces, merotopic spaces, linear topological spaces, etc.). It is seen from an investigation of the relations between these structures that some patterns are often repeated. This suggests attempting to describe these patterns generally. One of the results in this direction is introduced in the present paper. It concerns limits of presheaves, subobjects, quotients, modifications, etc. As an example, many constructions in topological spaces (**Top**) carry over from the category of sets **Ens** by means of the forgetful functor. E.g. we obtain the limit of a presheaf in **Top** first by constructing the limit in **Ens** and then by providing it with a suitable topology (similarly we obtain subspaces and quotients). This topology is the finest or the coarsest topology rendering the given mappings continuous, which can be again described by the forgetful functor (the property of being finer is equivalent to the existence of a continuous mapping, that is an identity in **Ens**). Hence generally, there is given in this paper a covariant functor $F : \mathscr{K} \to \mathscr{C}$ satisfying certain natural conditions (and then implying some results known in special structures).

Some further details and mainly applications of this general theory occur in References (for convenience of a reader the introduced theorems are not stated in full generality and the examples used are only illustrative). The first isolated results concerning this subject can already be found in the pre-war literature. Recently, a more comprehensive survey was pointed out e.g. by N. Bourbaki in [1] and by Z. Frolík, M. Katětov in [2].

Now, we shall introduce some terms used in the sequel. Let $\mathscr{K}$ be a category ($\mathscr{K}$ is the class of morphisms, and the class obj $\mathscr{K}$ of all objects in $\mathscr{K}$ is identified with the class of units). If $f \in \mathrm{Hom}_{\mathscr{K}} \langle X, Y \rangle$ then $X = \mathrm{D}f$, $Y = \mathrm{E}f$. A non-void family $\{f_i\}$ of morphisms from $\mathscr{K}$ is said to be projective if all the f_i have the same domain. I hope that it is clear what is meant by "dualization", and therefore we state definitions and theorems for the "projective case" only; the dual notions are "inductive".

I. In this part, let F be a covariant faithful functor from a category $\mathscr{K}$ into a category $\mathscr{C}$ (e.g. let F be the forgetful functor from **Top** into **Ens**). We can define a quasi-order on obj $\mathscr{K}$ thus:

$$X <_F Y \quad \text{if} \quad Ff = 1 \quad \text{for some} \quad f \in \mathrm{Hom}_{\mathscr{K}} \langle X, Y \rangle .$$

Definition 1. Let $\mathscr{C}'$ be a subcategory of $\mathscr{C}$. We shall say that a projective family $\{f_i\}$ in $\mathscr{K}$ is *projectively F-generating with respect to* $\mathscr{C}'$ if for any projective family $\{g_i\}$ in $\mathscr{K}$ with $\{\mathrm{E}g_i\} = \{\mathrm{E}f_i\}$, $\{Fg_i\} = \{Ff_i \circ \varphi\}$, $\varphi \in \mathscr{C}'$ there exists a $g \in \mathscr{K}$ such that $Fg = \varphi$ and $\{g_i\} = \{f_i \circ g\}$. We shall write $X = \langle F, \mathscr{C}' \rangle\text{-}\overleftarrow{\mathrm{Lim}}\,\{g_i\}$ if there is a projectively F-generating family $\{f_i\}$ with respect to $\mathscr{C}'$ such that $\{\mathrm{E}f_i\} = \{\mathrm{E}g_i\}$, $\{Ff_i\} = \{Fg_i\}$ and $X = \mathrm{D}f_i$.

Usually we have $\mathscr{C}' = \mathscr{C}$ (then Definition 1 is the Bourbaki definition of the initial structures), or $\mathscr{C}' = \mathrm{obj}\,\mathscr{C}$ (then we get the usual definition of e.g. weak topologies). Quite often the generation does not depend on $\mathscr{C}'$:

Theorem 1. *Let F fulfil the following condition:*

(α) *If* $\{f_i\}$ *is a projective family in* $\mathscr{K}$ *such that* $\{Ff_i\} = \{\varphi_i \circ \varphi\}$, $F^{-1}[\mathrm{E}\varphi] \neq \emptyset$ *then* $\{f_i\} = \{f_i' \circ f\}$ *where* $\{Ff_i'\} = \{\varphi_i\}$, $Ff = \varphi$ *and let* $\mathscr{C}' \supset \mathrm{obj}\,\mathscr{C}$, $\mathscr{C}'' \supset \mathrm{obj}\,\mathscr{C}$. *Then the projective F-generations with respect to* $\mathscr{C}'$ *and with respect to* $\mathscr{C}''$ *coincide.*

Among the consequences of the following two theorems there are statements like this: A topological space is projectively generated by mappings into spaces from a class $\mathscr{X}$ if and only if it can be embedded into a product of spaces from $\mathscr{X}$ and of an accrete space.

Theorem 2. *Let* $\{g_{ij} \mid j \in J_i\}$, $\{f_i \mid i \in I\}$ *be projective families in* $\mathscr{K}$ *such that* $\mathrm{E}f_i = \mathrm{D}g_{ij}$ *for every* i, j. *Let* $\mathscr{C}', \mathscr{C}'', \mathscr{C}'''$ *be subcategories of* $\mathscr{C}$.

(a) *If* $\{g_{ij} \circ f_i \mid j \in J_i,\ i \in I\}$ *is projectively F-generating with respect to* $\mathscr{C}'$ *then* $\{f_i\}$ *has the same property.*

(b) *If* $\{g_{ij} \mid j \in J_i\}$ *is projectively F-generating with respect to* $\mathscr{C}''$ *for each i,* $\{f_i\}$ *is projectively F-generating with respect to* $\mathscr{C}'$ *and if* $\mathscr{C}''' \subset \mathscr{C}''$, $Ff_i \circ \varphi \in \mathscr{C}''$ *for any* $\varphi \in \mathscr{C}'''$, *then* $\{g_{ij} \circ f_i\}$ *is projectively F-generating with respect to* $\mathscr{C}'''$.

Corollary. Let $\mathrm{obj}\,\mathscr{C} \subset \mathscr{C}'$, $X_i = \langle F, \mathscr{C}' \rangle\text{-}\overleftarrow{\mathrm{Lim}}\, f_i$ for each i. Then $X = \langle F, \mathrm{obj}\,\mathscr{C} \rangle\text{-}\overleftarrow{\mathrm{Lim}}\,\{f_i\}$ if and only if $X = \inf \{X_i\}$ (in $<_F$).

Theorem 3. *Let F have the following property*

(β) *If* $\{\varphi_i\}$ *is a projective family in* $\mathscr{C}$ *and if* $\{X_i\}$ *is a family in* $\mathrm{obj}\,\mathscr{K}$ *such that* $\{FX_i\} = \{\mathrm{E}\varphi_i\}$ *then there exists a projective family* $\{f_i\}$ *in* $\mathscr{K}$ *such that* $\{\mathrm{E}f_i\} = \{X_i\}$, $\{Ff_i\} = \{\varphi_i\}$.

Let H be a presheaf in $\mathscr{K}$. *Then* $\langle X, \{f_i\} \rangle$ *is the inverse limit of H if and only if* $X = \langle F, \mathscr{C} \rangle\text{-}\overleftarrow{\mathrm{Lim}}\,\{f_i\}$, $X = \mathrm{D}f_i$ *for each i and* $\langle FX, \{Ff_i\} \rangle$ *is the inverse limit of* $F \circ H$.

Corollary. Let $\mathscr{K}$ be product-admitting and let F fulfil (β). Assume that $\{f_i\}$ is a projective family in $\mathscr{K}$, $\langle Y, \{p_i\}\rangle$ is the product of $\{\mathrm{E}f_i\}$ and that f is the morphism from $\mathrm{D}f_i$ into Y such that $\{p_i \circ f\} = \{f_i\}$. Then $\{f_i\}$ is projectively F-generating with respect to $\mathscr{C}'$ if and only if f has the the same property.

Having defined subobjects in $\mathscr{C}$, the following definition of subobjects in $\mathscr{K}$ seems to be convenient:

Definition 2. Let $\mathscr{C}_0$ be a subcategory of $\mathscr{C}$. We shall say that a pair $\langle Y, f\rangle$ is an $(\langle F, \mathscr{C}'\rangle, \mathscr{C}_0)$-*subobject* of X if $f \in \mathrm{Hom}_{\mathscr{K}}\langle Y, X\rangle$ is projectively F-generating with respect to $\mathscr{C}'$ and $Ff \in \mathscr{C}_0$.

Let $\mathscr{C}_0 = \mathrm{E}\{\varphi \mid \varphi$ is a monomorphism in $\mathscr{C}$ such that $\varphi = \varphi_1 \circ \varphi_2$ with φ_2 an epimorphism and φ_1 a monomorphism implies that φ_2 is an equivalence$\}$, let $\mathscr{K}_0$ be defined similarly in $\mathscr{K}$, and let F be an S-functor (see Definition 4); then $\langle Y, f\rangle$ is an $(\langle F, \mathscr{C}\rangle, \mathscr{C}_0)$-subobject of X if and only if $f \in \mathrm{Hom}_{\mathscr{K}_0}\langle Y, X\rangle$.

II. Let $F : \mathscr{K} \to \mathscr{C}$, $G : \mathscr{K}_1 \to \mathscr{K}$ be covariant faithful functors (e.g. let $\mathscr{K}_1 =$ $=$ **Unif**, $\mathscr{K} =$ **Top**, $\mathscr{C} =$ **Ens** and F, G be the obvious functors).

Definition 3. The functor G is said to be *projectively F-stable with respect to* $\mathscr{C}'$ (or *projectively F-preserving with respect to* $\mathscr{C}'$) if $\langle F, \mathscr{C}'\rangle\text{-}\underleftarrow{\mathrm{Lim}}\,\{Gf_i\} =$ $= G[\langle F \circ G, \mathscr{C}'\rangle\text{-}\underleftarrow{\mathrm{Lim}}\,\{f_i\}]$ for any projective family $\{f_i\}$ in $\mathscr{K}_1$, whenever the left side (or the right side, respectively) of this equality exists.

The functor G is said to be $\mathscr{C}_0$-*hereditarily F-stable with respect to* $\mathscr{C}'$ (*or* $\mathscr{C}_0$-*hereditarily F-preserving with respect to* $\mathscr{C}'$) if for any $X \in \mathrm{obj}\,\mathscr{K}_1$ we have $\langle A, \varphi\rangle = \langle GY, Gf\rangle$ (where $\langle A, \varphi\rangle$ is an $(\langle F, \mathscr{C}'\rangle, \mathscr{C}_0)$-subobject of GX, and $\langle Y, f\rangle$ is an $(\langle F \circ G, \mathscr{C}'\rangle, \mathscr{C}_0)$-subobject of X) for each $\langle A, \varphi\rangle$ (or $\langle Y, f\rangle$, respectively).

The functor G is said to be *product-stable* (or *product-preserving*) if $\prod\{GX_i\} =$ $= G[\prod\{X_i\}]$ for any family $\{X_i\}$ in obj $\mathscr{K}_1$ whenever the left side (or the right side, respectively) of this equality exists. (The equalities in the last two definitions are up to an isomorphism.)

In the case that $\mathscr{K}_1$ is a subcategory of $\mathscr{K}$ and G is the embedding functor, we shall say that $\mathscr{K}_1$ has a property P in $\mathscr{K}$ rather than that G has the property P (e.g., $\mathscr{K}_1$ is product-stable in $\mathscr{K}$).

It is easy to obtain results concerning the relations between stability and preservation — e.g., the following proposition: Let the projective $F \circ G$-generation exist for any projective family in $\mathscr{K}_1$. Then G is projectively F-stable whenever it is projectively F-preserving.

Theorems concerning relations between projectivity, hereditariness and productivity are more useful. For the sake of clearness we shall state these only under conditions fulfilled by most continuity structures. We want for F and $F \circ G$ to satisfy conditions (α), (β), to have Lim $\{f_i\}$ for any projective family in $\mathscr{K}$ or $\mathscr{K}_1$, and the same for dual notions. We obtain an important class of S-functors including most of the forgetful functors of continuity structures:

Definition 4. The functor F is called an S-*functor* (and then $\mathscr{K}$ is called an S-*category with respect to* F) if the following conditions are fulfilled:

(1) $Ff = \varphi_1 \circ \varphi_2$ implies $f = f_1 \circ f_2$ where $Ff_i = \varphi_i$;

(2) if $\varphi \in \mathscr{C}$, $FX = \mathrm{E}\varphi$ (or $FX = \mathrm{D}\varphi$) then there exists an $f \in \mathscr{K}$ such that $Ff = \varphi$ and $X = \mathrm{E}f$ (or $X = \mathrm{D}f$, respectively);

(3) for each object A in $\mathscr{C}$ the class $F^{-1}[A] \cap \mathrm{obj}\, \mathscr{K}$ is the complete set in $<_F$;

(4) if $\{f_i\}$ is a family in $\mathscr{K}$ such that $Ff_i = \varphi$ for each i, then there is an $f \in \in \mathrm{Hom}_{\mathscr{K}} \langle \sup \{Df_i\}, \sup \{\mathrm{E}f_i\} \rangle$ with $Ff = \varphi$ and similarly for inf.

It may be useful to add a further condition on S-functors: if f is an equivalence in $\mathscr{K}$ such that $Ff = 1$ then $f = 1$. This condition implies the uniqueness of generation.

It follows from Theorem 1 that the generation by S-functors does not depend on $\mathscr{C}'$ if $\mathscr{C}' \supset \mathrm{obj}\, \mathscr{C}$. Hence we shall omit $\mathscr{C}'$ in this case.

Let F be an S-functor, $\mathscr{C}'$ a subcategory of $\mathscr{C}$ and $\mathscr{K}'$ a full subcategory of $F^{-1}[\mathscr{C}']$. Suppose that each object X of $F^{-1}[\mathscr{C}']$ has the lower or upper modification $\langle Y_X, f_X \rangle$ in $\mathscr{K}'$ such that $Ff_X = 1$. Then the restriction of F to $\mathscr{K}'$ and $\mathscr{C}'$ is an S-functor.

In the case that F and $F \circ G$ are S-functors, the preservation and the stability of G coincide. Therefore we shall say only that G is F-projective, productive or F-hereditary. The productivity and the hereditariness of a full subcategory $\mathscr{K}_1$ of $\mathscr{K}$ are then the known properties (e.g., $\mathscr{K}_1$ is F-hereditary in $\mathscr{K}$ if and only if each F-subobject in $\mathscr{K}$ of an object in $\mathscr{K}_1$ belongs to $\mathscr{K}_1$).

Throughout the remaining part F and $F \circ G$ are S-functors.

Theorem 4. *If G is F-projective then it is productive and F-hereditary. The converse is true if $\mathscr{C}_0$ is the class of all monomorphisms in $\mathscr{C}$, $\mathscr{C}$ is product-admitting and G preserves maximal objects (in $<_F$, $<_{F \circ G}$). The last condition is satisfied if $G[\mathrm{obj}\, \mathscr{K}_1] = \mathrm{obj}\, \mathscr{K}$.*

Proof of the converse statement. Let $X = F \circ G\text{-}\underleftarrow{\mathrm{Lim}}\, \{f_i\}$. We may suppose that one of the f_i is a monomorphism (e.g. into a maximal object greater than X). Then, by the Corollary to Theorem 3, X is an $F \circ G$-subobject of the product $\prod\{\mathrm{E}f_i\}$. Now, it is sufficient to use the preservation of products and subobjects by G, and the same Corollary again.

Next, we shall state some sufficient and necessary conditions for G to be F-projective.

Theorem 5. *The functor G is F-projective if it fulfils the condition* $(\boldsymbol{\beta})$ *(see Theorem 3). The converse is true if $G[\mathrm{obj}\, \mathscr{K}_1] = \mathrm{obj}\, \mathscr{K}$.*

Theorem 6. *Assume that $\mathscr{K}_1$ is a full subcategory of $\mathscr{K}$ and that G is the embedding of $\mathscr{K}_1$ into $\mathscr{K}$. Then each object X of $\mathscr{K}$ has an upper modification $\langle Y_X, f_X \rangle$ in $\mathscr{K}_1$ such that $Ff_X = 1$ if and only if $\mathscr{K}_1$ is F-projective in $\mathscr{K}$ and there* *exist* $Y'_X \in \mathrm{obj}\,\mathscr{K}_1$ *such that* $F[\mathrm{Hom}_{\mathscr{K}}\langle X, Z\rangle] \subset F[\mathrm{Hom}_{\mathscr{K}_1}\langle Y'_X, Z\rangle]$ *for any* $Z \in \mathrm{obj}\,\mathscr{K}_1$. *(The last condition is fulfilled if, for each* $X \in \mathrm{obj}\,\mathscr{K}$, *there is a* $Y'_X \in \mathrm{obj}\,\mathscr{K}_1$ *such that* $Y'_X <_F X$.*)*

The last two theorems deal with a particular kind of functors. The general case may be treated by decomposing G to a functor "onto" and to a full embedding, and applying the following theorem.

Theorem 7. *Let* $G_1 : \mathscr{K}_2 \to \mathscr{K}_1$ *be a covariant functor such that* $F \circ G \circ G_1$ *is an S-functor. Suppose that* $G_1[\mathrm{obj}\,\mathscr{K}_2] = \mathrm{obj}\,\mathscr{K}_1$ *and that* G *is a full embedding. Then* $G \circ G_1$ *is F-projective (productive, F-hereditary) if and only if* G *is F-projective (productive, F-hereditary) and* G_1 *is* $F \circ G$*-projective (productive,* $F \circ G$*-hereditary, respectively).*

One may obtain a number of theorems for continuity structures by combinations of Theorems 4,5 and 6. E.g., (a) The functor from **Unif** into **Top** is projective but it is not inductive (by Theorem 5); since it preserves sums, it cannot preserve quotients (by Theorem 4). (b) The category $\mathscr{K}$ of proximally fine uniformities is inductive in **Unif** (by Theorem 4); hence every uniform space has a lower modification in $\mathscr{K}$ (by Theorem 6).

It is seen from Theorem 6 that projectivity and inductivity is closely related to modifications. We shall state only the following proposition concerning the commuting of modifications and of generation: Let $\mathscr{K}$ be a subcategory of $\mathscr{K}_1$ such that each $X \in \mathrm{obj}\,\mathscr{K}_1$ has a lower modification $\langle Y_X, f_X\rangle$ in $\mathscr{K}$ (we may suppose that $f_X = 1$ if $X \in \mathrm{obj}\,\mathscr{K}$). Then there is defined in an obvious way a covariant functor $G : \mathscr{K}_1 \to \mathscr{K}$ assigning Y_X to X. Clearly G has property (β). Hence, if F and $F \circ G$ are S-functors, then G is F-projective; this means that the projective generation (and hence products, subobjects) commutes with lower modifications. This is e.g. the case when $\mathscr{K}_1 =$ **Top** and $\mathscr{K}$ is the category of convergence spaces.

The methods introduced in this paper can be used to construct functors and to obtain categorial characterizations of some continuity structures.

References

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