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IDENTIFYING HILBERT CUBES: GENERAL METHODS AND THEIR APPLICATION TO HYPERSPACES BY SCHORI AND WEST

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In the past several years new methods of identifying Hilbert cubes have been discovered, and they have been applied by R. M. Schori and the author to hyperspaces of some one-dimensional Peano continua. In particular, we have solved affirmatively the conjecture of M. Wojdyslawski [11] that the hyperspace 2^I of all non-void, closed subsets of the interval is a Hilbert cube, when topologized by the Hausdorff metric. I am informed by Professor Kuratowski that the conjecture was originally posed in the 1920's and was well-known to topologists in Warsaw and other places at that time. This report outlines some of these methods and the Schori-West proof of this conjecture.

Definitions and notation

Hilbert cubes will be denoted generically by Q and viewed primarily as infinite products of closed intervals. A space X will be called a *Hilbert cube factor* (Q-factor) if $X \times Q = Q$. The term "map" implies continuity, and homeomorphisms are all surjections. A map is a *near-homeomorphism* if it is the uniform limit of homeomorphisms. Additionally, a map $f : X \to Y$ *stabilizes to a near-homeomorphism* if $f \times id : X \times Q \to Y \times Q$ is a near-homeomorphism, "id" denoting the identity map. The *mapping cylinder* $M(f)$ of $f : X \to Y$ is the quotient space of $X \times I \cup Y$ under the relation identifying each $(x, 0)$ with $f(x)$, and if A is a closed subset of X, then $M(f, A)$ is the *reduced mapping cylinder of f relative to A* and is obtained from $X \times I \cup Y$ by identifying each point (a, t) of $A \times I$ with $f(a)$. The collapsing map $c : M(f) \to Y$ or $c : M(f, A) \to Y$ is that retraction induced on $M(f)$ or $M(f, A)$ by identifying each (x, t) of $X \times I$ with $f(x)$.

A closed subset A of a separable, metric absolute neighborhood retract X is said to have *property Z in X* if for each open subset U of X, the inclusion $U \setminus A \to U$ is a homotopy equivalence. (This is almost R. D. Anderson's definition [2] and is equivalent for Hilbert cube manifolds and Hilbert manifolds.)

Methods of identifying Hilbert cubes known prior to the second Symposium

Before 1966 there were four ways known to tell that a space is a Hilbert cube.

(1) *Infinite-dimensional, convex compacta of Hilbert spaces are Hilbert cubes.* (O.-H. Keller [5].)

(2) *Inverse limits of Hilbert cubes and near-homeomorphisms are Hilbert cubes.* (A specialization of a theorem of M. Brown [3] having the following as an immediate corollary.)

(2′) *An inverse limit of Hilbert cube factors and maps which stabilize to near-homeomorphisms is a Hilbert cube factor.*

(3) *Dendra are Hilbert cube factors.* (R. D. Anderson [1]. This is a solution of a problem in the *Scottish Book* posed by K. Borsuk. The proof was not published, but one appears in [8]. A. Szankowski [7] gave another proof that an infinite product of triods, i.e., spaces homeomorphic to the letter "T", is a Hilbert cube, which also answers Borsuk's problem.)

(4) *An infinite product* $\prod_{i=1}^{\infty} X_i$ *of non-degenerate Q-factors is a Hilbert cube if for each i and every positive number* ε *there are a homeomorphism* $g : X_i \times Q \to I \times Q$ *and a map* $f : X \to I$ *such that the I-coordinate of* $g(x, q)$ *is within* ε *of* $f(x)$ *for each point* (x, q) *of* $X_i \times Q$. (R. D. Anderson [1]. This was never published, either. A proof occurs in [8].)

New methods of identifying Hilbert cubes

Shortly after the second Symposium, Anderson established

(5) *A union of two Hilbert cubes intersecting in a third with property Z in each of the others is a Hilbert cube.* (Not explicitly stated, but a corollary to Theorem 10.1 of [2].)

This may be reformulated in a slightly weaker way as

(5′) *The union of two Hilbert cube factors intersecting in a third with property Z in each of the others is a Hilbert cube factor.*

I was able to generalize (5′) to read

(6) *The union of two Hilbert cube factors intersecting in a third which has property Z in one of the others is a Hilbert cube factor.* (Not explicit but equivalent to Theorem 5.1 of [8].) A corollary of (6) is

(7) *A compact contractible polyhedron is a Hilbert cube factor; a locally compact polyhedron is a Hilbert cube manifold factor and a Hilbert manifold factor.*

Also, I was able in [8] to lift completely the last hypothesis of (4), obtaining

(8) *An infinite product of non-degenerate Hilbert cube factors is a Hilbert cube.*

Two results have appeared on near-homeomorphisms of Hilbert cubes, necessary to the application of (2) and (2′). Both begin with (7) but point in different directions. The next, due to D. Curtis [4] is the definitive result in the simplicial category.

(9) *A simplicial surjection between locally finite simplicial complexes stabilizes to a near-homeomorphism if and only if each point-inverse is compact and contractible.*

My own result deals with mapping cylinders [9]:

(10) *The mapping cylinder $M(f)$ of any map between Hilbert cube factors is itself a Hilbert cube factor; moreover, the collapse $c : M(f) \to Y$ stabilizes to a near-homeomorphism.*

This allows the extension of (7) to cell-complexes.

(7′) *A compact, contractible cell-complex is a Hilbert cube factor; a locally compact cell-complex is a Hilbert cube manifold factor and a Hilbert manifold factor.*

Another consequence of (10) is a reduced version.

(11) *If $f : K \to Y$ is a map of a finite cell-complex to a Hilbert cube factor and A is any closed subset of K, then $M(f, A)$ is a Hilbert cube factor and its collapse $c : M(f, A) \to Y$ stabilizes to a near-homeomorphism.*

The next result is of a different sort and has been useful in work on hyperspaces [10].

(12) *If X is a Hilbert cube factor resulting from the compactification of a Hilbert cube manifold M by the addition of a second Hilbert cube factor $A = X \setminus M$ having property Z in X, then X is a Hilbert cube.*

Using (2′), (7), (8), (9) or (10), and (12), I was able to obtain a first result on hyperspaces [10]:

(13) *The hyperspace $C(D)$ of all subcontinua of a dendron D is a Hilbert cube factor which is a Hilbert cube if and only if the branch points of D are dense in D.* (The branch points of D are those separating it into more than two components.)

The current state of affairs is expressed in the next three results on hyperspaces, which are joint work of mine with R. M. Schori.

(14) *The hyperspace 2^I of all non-void closed subsets of the unit interval is a Hilbert cube.*

(15) *If Γ is any non-degenerate, connected, finite graph, then 2^{Γ} is a Hilbert cube.*

(16) *For any non-degenerate dendron D, 2^D is a Hilbert cube.*

2^I is a Hilbert cube, I: The strategy of the proof

I now show how Schori and I use the results discussed above to prove (14). We use inverse limits with (2), (2′), (8) and (10) as our starting point. We use the symbol 2^I_X to denote those members of 2^I containing X. If X is a finite set, we simply list its members as subscripts and begin by reducing the problem to that of showing that 2^I_{01}, those members of 2^I containing both 0 and 1, is a Hilbert cube. This is easy because Schori has shown in [6] that 2^I is the cone over the cone over 2^I_{01} and because (1) shows that the cone over a Hilbert cube is a Hilbert cube. We next derive (11) from (10) for technical convenience. After this the proof falls into three conceptual divisions:

(i) *2^I_{01} may be expressed as an inverse limit of spaces each homeomorphic to the infinite power of 2^I_{01}.*

(ii) *2^I_{01} is a Hilbert cube factor, so by (8) each space in the inverse limit of (i) is a Hilbert cube.*

(iii) *Each bonding map of the inverse limit of (i) is a near-homeomorphism, so by (2) 2^I_{01} is a Hilbert cube.*

The inverse system of (i) is as follows: For each positive integer n let $\sigma(n) = \{1, 1/n, 1/(n+1), \ldots\}$ and let $r_n : 2^I_{\sigma(n)} \to 2^I_{\sigma(n-1)}$ be defined by

$$r_n(A) = A \cup [m_n(A), m_n(A) + \delta_n(A)] \cup [M_n(A) - \delta_n(A), M_n(A)],$$

where

$$m_n(A) = \max(A \cap [0, 1/(n-1)]), \quad M_n(A) = \min(A \cap [1/(n-1), 1]),$$

and

$$\delta_n(A) = \min\{|1/(n-1) - a| \mid a \in A\}.$$

(17) $2^I_{01} = \operatorname{inv}\lim\{2^I_{\sigma(n)}, r_n\}$.

(18) *The natural function ψ: $2^I_{\sigma(n)} \to 2^{[1/n,1]}_{1/n\,1} \times 2^{[1/(n+1),1/n]}_{1/(n+1)\,1/n} \times \ldots$ is a homeomorphism.*

These establish (i).

2^I is a Hilbert cube, II: 2^I_{01} is a Hilbert cube factor

We use another inverse limit for 2^I_{01} in order to establish that it is a Q-factor: For each real number $t \geqq 1$, let $F_t : 2^I_{01} \to 2^I_{01}$ be the map sending each set A to its closed $1/t$-neighborhood in I and let $B_n = F_n(2^I_{01})$. Let $f_n = F_{n(n-1)} \mid B_n$. Then $F_{n-1} = f_n F_n$ and

(19) $2^I_{01} = \operatorname{inv} \lim \{B_n, f_n\}$.

Observe that B_n is composed of all members A of 2^I_{01} such that for some m, $A = \bigcup_{k=1}^{m} [a_{2k-1}, a_{2k}]$ where $a_1 = 0$, $a_{2m} = 1$, $a_{2k+1} > a_{2k}$, and $a_{2k} - a_{2k-1} \geqq 2/n$ unless $k = 1$ or m, in which case $a_{2k} - a_{2k-1} \geqq 1/n$.

(20) *Each B_n is a Hilbert cube factor.*

This may be proved in two ways. First, B_n is contractible because the maps $\{F_t \mid B_n\}_{t \geq 1}$ form a homotopy joining the constant map F_1 and the identity $(= \lim_{t \to \infty} F_t)$. Because B_n is a polyhedron, (7) applies. The second way is of use in proving (23) below. Let

$$B_n^m = \{A \in B_n \mid A \text{ has no more than } m \text{ components}\},$$
$$C_n^m = \{A \in B_n \mid A \text{ has exactly } m \text{ components}\}.$$

If φ is the map of C_n^m to R^{2m} sending each set A to the sequence $(a_1, \ldots, a_{2m})$ of its end-points, then φ embeds C_n^m in a $(2m - 2)$-simplex Δ of R^{2m} and its image misses exactly the union X of every second face of Δ. Moreover, φ^{-1} extends naturally to a map of Δ carrying X into B_n^{m-1}. Because X is homeomorphic to I^{2m-3} and C_n^m is open in B_n^m, we have

(21) *B_n^m is homeomorphic to the mapping cylinder of $\varphi^{-1} \mid X : X \to B_n^{m-1}$.*

Because $B_n^1 = \{I\}$, we have $B_n \times Q = Q$ by induction using (10). We prove each f_n stabilizes to a near-homeomorphism in (22) and (23): Let, for $0 \leqq t \leqq 1$, $\lambda_t : B_n \to B_n$ be given by

$$\lambda_t\Big(\bigcup_{k=1}^{m} [a_{2k-1}, a_{2k}]\Big) = \bigcup_{k=1}^{m} [a_{2k-1}^t, a_{2k}^t],$$

where

(i) $a_i^t = t a_i^1 + (1 - t)\, a_i$,

(ii) $a_{2k}^1 = a_{2k} + 1/n$ unless $a_{2k+1} - a_{2k} \leqq 2/n$, in which case we set $a_{2k}^1 = 1/2(a_{2k+1} + a_{2k})$, and

(iii) $a_{2k+1}^1 = a_{2k+1} - 1/n$ unless $a_{2k+1} - a_{2k} \leqq 2/n$, in which case we set $a_{2k+1}^1 = 1/2(a_{2k+1} + a_{2k})$.

(22) *For $0 < t < 1$, λ_t is an embedding of B_n in itself with image containing B_{n-1}. Moreover, $B_{n-1} \cup \lambda_t(B_n^m)$ is a reduced mapping cylinder of a map from a $(2m - 3)$-cell to $B_{n-1} \cup \lambda_t(B_n^{m-1})$.*

This is proved using the map $\varphi : C_n^m \to R^{2m}$ to obtain a geometric representation. If $c_{mt} : B_{n-1} \cup \lambda_t(B_n^m) \to B_{n-1} \cup \lambda_t(B_n^{m-1})$ is the collapse of this mapping cylinder and $g_{nt} = c_{2t} c_{3t} \dots c_{nt} \lambda_t : B_n \to B_{n-1}$, then

(23) *The mapping cylinder structures of* (22) *may be chosen so that as t approaches* 1, *g_{nt} approaches f_n uniformly. As each g_{nt} stabilizes to a near-homeomorphism, so does f_n.*

Applying (2′) to (21) and (23), we have II.

2^I is a Hilbert cube, III: r_n is a near-homeomorphism

In the representation ψ of $2^I_{\sigma(n)}$ as an infinite power of 2^I_{01}, r_n appears as the stabilization of the map $r^{1/n} : 2^{[1/n,1]}_{1/n1} \to 2^{[1/n,1]}_{1/n1/(n-1)1}$ defined by the same formula. Therefore, by using (2) it is sufficient to establish

(24) *For any t in* (0, 1), *the retraction $r^t : 2^I_{01} \to 2^I_{0t1}$ defined in strict analogy to r_n stabilizes to a near-homeomorphism.*

We achieve (24) by using $\{B_n, f_n\}$ and

(25) *Let* $X = \operatorname{inv\,lim} \{X_n, f_n\}$ *and* $Y = \operatorname{inv\,lim} \{Y_n, g_n\}$ *be compact metric spaces, and for each n let $h_n : X_n \to Y_n$ be a map such that $g_n h_n = h_{n-1} f_n$. The induced map* $h = \operatorname{inv\,lim} \{h_n\} : X \to Y$ *is a near-homeomorphism if each f_n and each h_n are, so h stabilizes to a near-homeomorphism if each f_n and each h_n do.*

We express r^t as a limit map in the inverse sequence $\{B_n, f_n\}$ as follows: Let $H_n = B_n \cap 2^I_{t-1/n}$, $K_n = H_n \cap 2^I_{t+1/n}$, and $L_n = F_n(2^I_{0t1}) = B_n \cap 2^I_{[t-1/n,t+1/n]}$. Also, let $s_n : K_n \to L_n$ be the map defined by the formula $A \to A \cup [t - 1/n, t + 1/n]$, and let $h_n = s_n r^{t+1/n} r^{t-1/n} \mid B_n$.

(26) $2^I_{0t1} = \operatorname{inv\,lim} \{L_n, f_n \mid L_n\}$, $f_n h_n = h_{n-1} f_n$ *and* $r^t = \operatorname{inv\,lim} \{h_n\}$.

After this, all that remains is to prove

(27) *Each of $r^{t-1/n} \mid B_n$, $r^{t+1/n} \mid H_n$, and s_n stabilizes to a near-homeomorphism.*

We prove (27) by using (11) inductively on a reduced mapping cylinder construction of B_n, H_n, and K_n from H_n, K_n, and L_n, respectively. The sequence (25)–(27) establishes (26), hence III, so the proof is completed.

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