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ON COMPLETIONS OF CONVERGENCE COMMUTATIVE GROUPS

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In this paper the notions of Cauchy sequences and completeness for convergence commutative groups are introduced. For any such group L a completion L_1 is constructed containing L as a dense subspace and a subgroup. Finally, examples are given which show that L can have more than one completion.

1.

Let L be a point set, $\mathfrak{L}$ a convergence and λ a convergence closure for L (λA is the set of all $\lim x_n \in L$ such that $\bigcup x_n \subset A$). We have a convergence space $(L, \mathfrak{L}, \lambda)$. Instead of $\lim x_n = x$ we sometimes write $(\{x_n\}, x) \in \mathfrak{L}$ or $\mathfrak{L}$-$\lim x_n = x$. Using the transfinite induction we define successive closures $\lambda^\xi A$ in the following manner

$$\lambda^0 A = A\,, \quad \lambda^1 A = \lambda A\,, \quad \lambda^\xi A = \bigcup_{\eta<\xi} \lambda\lambda^\eta A\,.$$

The closure operator λ^{ω_1} is a topology for L. A subset $A \subset L$ is λ-closed if $\lambda A = A$. A subset $B \subset L$ is λ^{ω_1}-dense in L provided that $\lambda^{\omega_1} B = L$.

Definition 1. Let $(L_1, \mathfrak{L}_1, \lambda_1)$ be a convergence space. Define $(\{x_n\}, x) \in \mathfrak{L}_1^*$ whenever for each subsequence $\{x_{n_i}\}$ of $\{x_n\}$ there is a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $(\{x_{n_{i_k}}\}, x) \in \mathfrak{L}_1$. We say that $\mathfrak{L}_1^*$ is a *star convergence*[1]) *of the convergence* $\mathfrak{L}$.

It is easy to prove that $\mathfrak{L}_1 \subset \mathfrak{L}_1^*$, $(\mathfrak{L}_1^*)^* = \mathfrak{L}_1^*$ and $\lambda_1^* = \lambda_1$.

Let $(L, \mathfrak{L}, \lambda)$ be a convergence space and assume that $\mathfrak{L}$ is a star convergence. Let $+$ be a commutative group operation on L. If $\lim x_n = x$ and $\lim y_n = y$ implies that $\lim (x_n - y_n) = x - y$ then we have [3] (see also [4]) a convergence commutative group (abbr. a cc group). It will be denoted $(L, \mathfrak{L}, \lambda, +)$.

Definition 2. Let $(L, \mathfrak{L}, \lambda, +)$ be a cc group. The collection of all pairs $(\{x_n\}, \{y_n\})$, where $\{x_n\}$, $\{y_n\}$ are sequences of points of L such that $\lim (x_{i_n} - y_{j_n}) = 0$ for all subsequences $\{i_n\}$ and $\{j_n\}$ of $\{n\}$ will be denoted ϱ. A sequence $\{x_n\}$ of points of L is called a *Cauchy sequence* (in L) if $(\{x_n\}, \{x_n\}) \in \varrho$.

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[1]) P. Urysohn [5] calls $\mathfrak{L}_1^*$ the convergence a posteriori. $\mathfrak{L}_1^*$ is sometimes called the maximal or largest convergence [1], [2].

Lemma 1. *If* $(\{x_n\}, \{y_n\}) \in \varrho$ *then both* $\{x_n\}$ *and* $\{y_n\}$ *are Cauchy sequences.*

Lemma 2. *Each subsequence of a Cauchy sequence is a Cauchy sequence.*

Lemma 3. *If* $(\{x_n\}, x) \in \mathfrak{L}$ *then* $\{x_n\}$ *is a Cauchy sequence.*

Lemma 4. *Let* $\{x_n\}$ *be a sequence and* $\{y_n\}$ *a Cauchy sequence. Then* $(\{x_n\}, \{y_n\}) \in \varrho$ *iff* $\lim (x_n - y_n) = 0$.

Lemma 5. *If* $(\{x_n\}, \{x'_n\}) \in \varrho$ *and* $(\{y_n\}, \{y'_n\}) \in \varrho$ *then* $(\{x_n - y_n\}, \{x'_n - y'_n\}) \in \varrho$.

Lemma 6. *If* $\lim x_n = x$, *then* $\lim y_n = x$ *iff* $(\{x_n\}, \{y_n\}) \in \varrho$.

The proofs of Lemmas 1–6 are easy and only hints are given here:

Lemma 1. $0 = \lim (x_{i_n} - y_{j_n}) - \lim (x_{j_n} - y_{j_n}) = \lim (x_{i_n} - x_{j_n})$.
Lemma 2. Evident.
Lemma 3. $\lim x_{i_n} = x = \lim x_{j_n}$ implies $\lim (x_{i_n} - x_{j_n}) = 0$.
Lemma 4. $\lim (x_n - y_{j_n}) - \lim (y_n - y_{j_n}) = \lim (x_n - y_n)$.
Lemma 5. $\lim (x_{i_n} - x'_{j_n}) - \lim (y_{i_n} - y'_{j_n}) = \lim ((x_{i_n} - y_{i_n}) - (x'_{j_n} - y'_{j_n}))$.
Lemma 6. $\lim x_n = x = \lim y_n$ implies $0 = \lim (x_{i_n} - x) + \lim (x - y_{j_n}) = \lim (x_{i_n} - y_{j_n})$. Now, if $\lim x_n = x$ and $(\{x_n\}, \{y_n\}) \in \varrho$ then $0 = \lim (x - x_n) + \lim (x_n - y_n) = \lim (x - y_n) = x - \lim y_n$.

From Definition 2 and from Lemma 1 it easily follows that ϱ is an equivalence relation on the set of all Cauchy sequences. The class of all Cauchy sequences which are equivalent to a Cauchy sequence $\{x_n\}$ will be denoted $[\{x_n\}]$. Evidently, $\lim x_n = x$ iff $[\{x_n\}] = [\{x\}]$, $\{x\}$ being the constant sequence.

Definition 3. A subset A of a cc group $(L, \mathfrak{L}, \lambda, +)$ is called *complete* provided that each Cauchy sequence $\{x_n\}$, $x_n \in A$, converges to a point of A.

Lemma 7. *A subset* A *of a complete cc group* $(L, \mathfrak{L}, \lambda, +)$ *is complete iff it is* λ*-closed.*

The easy proof is omitted.

Let $(L_\iota, \mathfrak{L}_\iota, \lambda_\iota, +_\iota)$, $\iota \in I$ be cc groups. Denote $L = \times\{L_\iota; \iota \in I\}$ the Cartesian product of L_ι, $\mathfrak{L}$ the coordinatewise convergence on L, λ the convergence closure for L induced by $\mathfrak{L}$ and $+$ the coordinatewise group operation on L. Then $\mathfrak{L}$ is a star convergence [2] and we have a Cartesian convergence commutative group $(L, \mathfrak{L}, \lambda, +)$.

Lemma 8. *Let* $(L, \mathfrak{L}, \lambda, +)$ *be a Cartesian cc group defined by cc groups* $(L_\iota, \mathfrak{L}_\iota, \lambda_\iota, +_\iota)$, $\iota \in I$. *Then* $\{(x^n_\iota)\}_{n=1}^\infty$ *is a Cauchy sequence in* $(L, \mathfrak{L}, \lambda, +)$ *iff* $\{x^n_\iota\}_{n=1}^\infty$ *is a Cauchy sequence in* $(L_\iota, \mathfrak{L}_\iota, \lambda_\iota, +_\iota)$ *for each* $\iota \in I$.

The proof is evident.

Lemma 9. *Let* $(L_\iota, \mathfrak{L}_\iota, \lambda_\iota, +_\iota)$, $\iota \in I$, *be complete cc groups. Let* $(L, \mathfrak{L}, \lambda, +)$ *be their Cartesian cc group. Let G be a subgroup of the group* $(L, +)$. *Then* $\lambda^{\omega_1}G$ *is the smallest complete convergence group containing G as a subgroup.*

Proof. $\lambda^{\omega_1}G$ is the smallest λ-closed subgroup of L containing G as a subgroup. L is complete by Lemma 8. Hence the assertion instantly follows from Lemma 7.

2.

Definition 4. Let $(L, \mathfrak{L}, \lambda, +)$ be a cc group. A cc group $(L_1, \mathfrak{L}_1, \lambda_1, +)$ is called a *completion* of $(L, \mathfrak{L}, \lambda, +)$ if it is complete and such that L is a $\lambda_1^{\omega_1}$-dense subspace of $(L_1, \mathfrak{L}_1, \lambda_1)$ and a subgroup of $(L_1, +)$.

Theorem 1. *Each cc group* $(L, \mathfrak{L}, \lambda, +)$ *has a least one completion* $(L_1, \mathfrak{L}_1, \lambda_1, +)$.

The proof of Theorem 1 is divided into two parts **A** and **B**. In the first part a cc group $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ is constructed such that L is a dense subspace of L_1 and a subgroup of L_1. In the second part a definition and a lemma are given and it is proved that $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ is complete.

A. Let $(L, \mathfrak{L}, \lambda, +)$ be a cc group. Let X be a point set of power $|X| > 2^{|L|}$ containing L as a subset. Let g be a one-to-one map on the set of all classes $[\{x_n\}]$ of Cauchy sequences into X such that $g([\{x_n\}])$ is a point x of L iff $(\{x_n\}, x) \in \mathfrak{L}$. Denote L_1 the set of all points $g([\{x_n\}])$ in X. Then $L \subset L_1 \subset X$. Now define a binary operation $\dotplus$ on L_1:

Definition 5. Let $z = g([\{x_n\}])$ and $t = g([\{y_n\}])$ be points of L_1. By Lemma 5, $\{x_n + y_n\}$ is a Cauchy sequence in L. We put $z \dotplus t = g([\{x_n + y_n\}])$.

In view of Lemma 5 the operation $\dotplus$ does not depend on representatives of classes.

If $x \in L$, $y \in L$ then $x \dotplus y = g([\{x\}]) + g([\{y\}]) = g([\{x + y\}]) = x + y$. Consequently, we may write $+$ instead of $\dotplus$ on L_1. If $\{x_n\}$ is a Cauchy sequence in L, then $-g([\{x_n\}]) = g([\{-x_n\}])$.

Statement 1. $(L_1, +)$ *is a commutative group containing* $(L, +)$ *as a subgroup.*

The proof follows instantly from Definition 5.

Definition 6. Let $\mathfrak{L}_1$ be the set of all pairs $(\{z_n\}, z)$, $z_n \in L_1$, $z \in L_1$ such that there is a Cauchy sequence $\{x_m\}$, $x_m \in L$, with the property[2]) $z - z_n = g([\{x_m\}]) - x_n$.

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[2]) If $\{y_n\}$, $y_n \in L$, is another Cauchy sequence then $g([\{x_n\}]) - x_n = g([\{y_n\}]) - y_n$ iff $x_n - y_n$ is a constant point in L for each n. It follows that $(\{z_n\}, z) \in \mathfrak{L}_1$ can be defined by more than one Cauchy sequence in L.

Statement 2. $\mathfrak{L}_1$ *is a convergence on* L_1.

Proof. First prove that $(\{z_n\}, z') \in \mathfrak{L}_1$, $(\{z_n\}, z'') \in \mathfrak{L}_1$ implies $z' = z''$. As a matter of fact, let $\{x_m\}$ and $\{y_m\}$ be Cauchy sequences in L such that $z' - z_n = a - x_n$, $z'' - z_n = b - y_n$ where $a = g([\{x_m\}])$ and $b = g([\{y_m\}])$. Denote $c = a - b + (z'' - z')$. Then $y_n = x_n - c$. It follows $c \in L$ and so $b = g([\{x_m - c\}]) = a - c = b - (z'' - z')$. Hence $z'' - z' = 0$.

Evidently $(\{z\}, z) \in \mathfrak{L}_1$ for each $z \in L_1$ and $(\{z_n\}, z) \in \mathfrak{L}_1$ implies that $(\{z_{n_i}\}, z) \in \mathfrak{L}_1$ for each subsequence $\{n_i\}$ of $\{n\}$. It follows that $\mathfrak{L}_1$ is a convergence on L_1.

Statement 3. $(\{z_n\}, z) \in \mathfrak{L}_1$ *and* $(\{t_n\}, t) \in \mathfrak{L}_1$ *implies* $(\{z_n - t_n\}, z - t) \in \mathfrak{L}_1$.

Proof. Let $z - z_n = a - x_n$ and $t - t_n = b - y_n$ where $a = g([\{x_n\}])$ and $b = g([\{y_n\}])$. Then $z - t - (z_n - t_n) = a - b - (x_n - y_n) = g([\{x_n - y_n\}]) - (x_n - y_n)$. Hence $\mathfrak{L}_1\text{-lim}\,(z_n - t_n) = (z - t)$ by Definition 6.

Let us notice that $\mathfrak{L}_1$ need not be a star convergence on L_1. It is easy to see that $\mathfrak{L}_1 = \mathfrak{L}_1^*$ iff $\mathfrak{L}_1 = \mathfrak{L}$, i.e., iff $(L, \mathfrak{L}, \lambda, +)$ is a complete cc group.

Statement 4. *Let* $\mathfrak{L}_1^*$ *be a star convergence of the convergence* $\mathfrak{L}_1$. *Then* $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ *is a cc group.*

Proof. Let $(\{a_n\}, a) \in \mathfrak{L}_1^*$ and $(\{b_n\}, b) \in \mathfrak{L}_1^*$ and let $\{a_{n_i} - b_{n_i}\}$ be any subsequence of $\{a_n - b_n\}$. Then, by Definition 1, there is a subsequence $\{n_{i_k}\}_{k=1}^{\infty}$ of $\{n_i\}_{i=1}^{\infty}$ such that $(\{a_{n_{i_k}}\}, a) \in \mathfrak{L}_1$ and $(\{b_{n_{i_k}}\}, b) \in \mathfrak{L}_1$. Consequently, from Statement 3 it follows that $(\{a_{n_{i_k}} - b_{n_{i_k}}\}, a - b) \in \mathfrak{L}_1$. Hence $(\{a_n - b_n\}, a - b) \in \mathfrak{L}_1^*$ by Definition 1.

Now, we are going to prove

Lemma 10. *If* $(\{x_n\}, 0) \in \mathfrak{L}_1^*$, $x_n \in L$, *then* $(\{x_n\}, 0) \in \mathfrak{L}$.

Proof. Let $\{x_{n_i}\}$ be a subsequence of $\{x_n\}$. Since $\mathfrak{L}_1^*$ is a star convergence, there is a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $(\{x_{n_{i_k}}\}, 0) \in \mathfrak{L}_1$. By Definition 6 there is a Cauchy sequence $\{y_m\}_{m=1}^{\infty}$, $y_m \in L$, in L such that $-x_{n_{i_k}} = g([\{y_m\}]) - y_k$. It follows that $g([\{y_m\}]) \in L$ and consequently $\mathfrak{L}\text{-lim}\,(g([\{y_m\}]) - y_k) = 0$. Hence $\mathfrak{L}\text{-lim}\,(-x_{n_{i_k}}) = 0$. Since $\mathfrak{L} = \mathfrak{L}^*$ we have $\mathfrak{L}\text{-lim}\, x_n = 0$.

From Lemma 10 immediately follows

Statement 5. *L is a subspace of* L_1.

From Statements 1–5 it follows that $(L_1, \mathfrak{L}_1^*, \lambda_1, +)$ is a cc group containing $(L, \mathfrak{L}, \lambda. +)$ as a subgroup and a subspace such that $\lambda_1^{\omega} L = \lambda_1 L = L_1$.

B. Definition 7. Points z, t of L_1 are called *equivalent* provided that $z - t \in L$. The class of all points of L_1 which are equivalent to z will be denoted $[z]$.

Lemma 11. *If* $(\{z_n\}, 0) \in \mathfrak{L}_1$, $z_n \in L_1$, *then* $z_n \in [z_1]$ *for each natural* n.

Proof. By Definition 6 there is a Cauchy sequence of points $x_m \in L$ such that $-z_n = g([\{x_m\}]) - x_n$. Hence $z_1 - z_n = x_1 - x_n \in L$ for each n.

Statement 6. $(L_1, \mathfrak{L}_1, \lambda_1, +)$ *is a complete cc group.*

Proof. Let $\{a_n\}$ be a Cauchy sequence in L_1. Then, by Definition 2, $(a_{i_n} - a_{j_n}, 0) \in \in \mathfrak{L}_1^*$, $\{i_n\}$ and $\{j_n\}$ being any subsequences of $\{n\}$. Consider two cases:

(1) There is a subsequence $\{b_n\}$ of $\{a_n\}$ such that $i \neq j$ implies $b_i \notin [b_j]$. Then $\{b_n\}$ is one-to-one. Construct a subsequence $\{b_{i_n}\}$ of $\{b_n\}$ as follows: Put $i_1 = 2$. Suppose we have just chosen $k - 1$ naturals i_m such that $2^{m-1} < i_m \leqq 2^m$ and that no two distinct members of the sequence $\{b_m - b_{i_m}\}_{m=1}^{k-1}$ are equivalent. Consider the sequence $\{b_k - b_n\}_{n=2^{k-1}+1}^{2^k}$. It contains 2^{k-1} points $b_k - b_n$. Because $2^{k-1} > > k - 1$ there is a natural i_k, $2^{k-1} < i_k \leqq 2^k$ such that the point $b_k - b_{i_k}$ fails to be equivalent to any of the points $b_m - b_{i_m}$ (otherwise there would be two indices n_1, n_2, $2^{k-1} < n_1 < n_2 \leqq 2^k$ such that $b_k - b_{n_1} \in [b_k - b_{n_2}]$, i.e., $b_{n_1} \in [b_{n_2}]$, which is impossible). In such a way we have a sequence $\{b_n - b_{i_n}\}_{n=1}^{\infty}$ no two distinct members of which are equivalent. By Lemma 11, no its subsequence $\mathfrak{L}_1$-converges to 0. On the other hand, $\{b_n\}$ is a Cauchy sequence, by Lemma 2. Hence $(\{b_n - b_{i_n}\}, 0) \in \in \mathfrak{L}_1^*$. In view of Definition 1 we have a contradiction.

It follows that the case (1) cannot occur.

(2) There exist a point $z \in L$ and a subsequence $\{c_n\}$ of $\{a_n\}$ such that $c_n \in [z]$ for each n. Hence $c_n = z + r_n$, r_n being suitable points of L. Since $\{c_n\}$ is a Cauchy sequence we have $(\{c_{i_n} - c_{j_n}\}, 0) \in \mathfrak{L}_1^*$ for any subsequences $\{i_n\}$ and $\{j_n\}$ of $\{n\}$. Consequently, $(\{r_{i_n} - r_{j_n}\}, 0) \in \mathfrak{L}_1^*$. According to Lemma 10 and Definition 2, $\{r_n\}$ is a Cauchy sequence in L. Denote $c = g([\{r_n\}])$. Since $c - r_n = g([\{r_n\}]) - r_n$ we have $(\{r_n\}, c) \in \mathfrak{L}_1 \subset \mathfrak{L}_1^*$, by Definition 5, and so $(\{c_n\}, c + z) \in \mathfrak{L}_1^*$, by Statement 4. Hence $\mathfrak{L}_1^*$-lim $a_n = c + z$, by Lemma 6.

3.

Examples. Let X be a non void point set. Let $\mathfrak{L}$ denote the usual set convergence on the system $\boldsymbol{X}$ of all subsets of X. Then $(\boldsymbol{X}, \mathfrak{L}, \lambda, \div)$ is a complete cc group. As a matter of fact, if $\{A_n\}$, $A_n \in \boldsymbol{X}$, is a Cauchy sequence then Lim inf A_n = Lim sup A_n. Otherwise, there would be two subsequences $\{i_n\}$ and $\{j_n\}$ of $\{n\}$ and a point $x \in \in A_{i_n} \div A_{j_n}$ for each n. This is a contradiction. From Lemma 7 it follows that each ring of sets $\boldsymbol{R} \subset \boldsymbol{X}$ considered as a cc group has a completion in $\boldsymbol{X}$, viz. the sigma ring $\boldsymbol{S}(\boldsymbol{R})$ over $\boldsymbol{R}$, because $\lambda^{\omega_1}\boldsymbol{R} = \boldsymbol{S}(\boldsymbol{R})$.

Let $\mathscr{F}$ be the class of all real valued functions on X. Let $\mathfrak{L}'$ be the convergence on $\mathscr{F}$ at each point. From Lemma 9 it follows that $(\mathscr{F}, \mathfrak{L}', \lambda', +)$ is a complete cc group. Now, if X is the real line R_1, $\mathscr{C}$ the class of all continuous functions on R_1 and $\mathscr{B}$ the class of all Baire functions then $\lambda^{\omega_1}\mathscr{C} = \mathscr{B}$. Hence $\mathscr{B}$ is a completion of $\mathscr{C}$, by Lemma 7.

A cc group can have several completions which are not homeomorphic. This will be illustrated by following examples.

Let R_1 be the set of all real numbers and R the set of all rational numbers. Let u_1 and u be the usual topologies for R_1 and R. Then the usual topological group $(R, \mathfrak{R}, u, +)$ of rationals is a cc group. It has two different completions. One of them is $(R_1, u_1, +)$ and the other, by Theorem 1, is the cc group $(R_1, \mathfrak{R}_1^*, \lambda_1, +)$ of real numbers the closure of which differs from the usual closure for reals. From Lemma 11 we deduce that no subsequence of the sequence $\{n^{-1}\sqrt{2}\}$ $\mathfrak{R}_1^*$-converges to 0. Hence $0 \in u_1 \bigcup_n n^{-1}\sqrt{2} - \lambda_1 \bigcup_n n^{-1}\sqrt{2}$.

Let $\boldsymbol{F}$ be the class of all finite subsets of an infinite set X. Then $(\boldsymbol{F}, \mathfrak{L}, \lambda, \div)$ is a cc group. There are two completions of $\boldsymbol{F}$, both consisting of all countable subsets of X. The convergence of the first completion is the usual set convergence whereas the convergence $\mathfrak{L}_1^*$ of the other completion from Theorem 1 is different from the usual set convergence. Notice that, by Lemma 11, the sequence of disjoint infinite sets has no subsequence $\mathfrak{L}_1^*$-converging to $\emptyset$.

Also the cc group $\mathscr{C}$ consisting of all continuous functions $f(x)$, $x \in R_1$, has two different completions, one being the cc group $\mathscr{B}$ of all Baire functions and the other is a subgroup of $\mathscr{B}$ with a special convergence $\mathfrak{L}_1^*$ at each point defined in Theorem 1 (Definition 6).

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