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ON THE SATURATION OF A TOPOLOGICAL PARTIAL ALGEBRA WITH RESPECT TO A CONGRUENCE RELATION

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This is a report on a paper [4] being in preparation, in which extensions of a given topological partial algebra to topological (full) algebras are discussed. The notion of the saturation of a topological partial algebra with respect to a congruence relation serves as a mean to consider topological quotient partial algebras and such extensions within a common framework.

1. Let $\mathfrak{A} = (A, (f_\gamma)_{\gamma<\beta})$ be a partial algebra (for the algebraic terminology, see Grätzer [1], especially p. 79) with the base set A and the n_γ-ary partial operations f_γ on A ($0 < n_\gamma < \omega$, β an ordinal >0), furthermore, let Θ be a congruence relation of $\mathfrak{A}$ (see [1], p. 82). For each set $M \subseteq A$, the set $\{x \mid x \in A$ and, for some $y \in M$, $x\Theta y\}$ is said to be the *Θ-saturation of M*. Define, for each $\gamma < \beta$ and each $(X_0, \ldots, \ldots, X_{n_\gamma-1}) \in (\mathfrak{P}A)^{n_\gamma}$ ($=n_\gamma$-th Cartesian power of the power set $\mathfrak{P}A$ of A), the set $({}^\Theta f_\gamma)(X_0, \ldots, X_{n_\gamma-1})$ to be the Θ-saturation of the set $f_\gamma(X_0 \times \ldots \times X_{n_\gamma-1})$. Then, the algebra $(\mathfrak{P}A, ({}^\Theta f_\gamma)_{\gamma<\beta})$ with the n_γ-ary operations ${}^\Theta f_\gamma$ is called the *Θ-saturation ${}^\Theta\mathfrak{A}$ of $\mathfrak{A}$.*

Examples.

1. Let $\Theta = A \times A$. Then, for each $\gamma < \beta$, $({}^\Theta f_\gamma)(X_0, \ldots, X_{n_\gamma-1})$ is equal to A if $(\mathscr{D}f_\gamma) \cap (X_0 \times \ldots \times X_{n_\gamma-1}) \neq \emptyset$, and equal to $\emptyset$ otherwise. ($\mathscr{D}f_\gamma$ denotes the domain of f_γ.)

2. Let $\Theta = id_A$ ($=$ identical mapping on A). Then, for each $\gamma < \beta$, $({}^\Theta f_\gamma)(X_0, \ldots, X_{n_\gamma-1}) = f_\gamma(X_0 \times \ldots \times X_{n_\gamma-1})$, and, in this case, we will call ${}^\Theta\mathfrak{A}$ the *power* $\mathfrak{P}\mathfrak{A}$ *of* $\mathfrak{A}$ (not to be mixed up with the power set of the set $\mathfrak{A}$).

As an immediate consequence of the definitions, we have

Proposition 1. *The quotient partial algebra* $\mathfrak{A}/\Theta$ (see [1], p. 82) *is the relative (partial) subalgebra* (see [1], p. 80) *of* $\mathfrak{A}$ *with* A/Θ *as the base set.*

2. Let (A, τ) be a topological space. The *(first) power* $\mathfrak{P}\tau$ *of the topology* τ is defined to be that topology on the power set $\mathfrak{P}A$ of A which is induced by the limit operator ("Limesoperator")

$$\mathfrak{P} \circ (\lim\inf_\tau)_{\Phi_0(\mathfrak{P}A)}$$

(= composition of the mapping "limit inferior" $\lim\inf_\tau$, restricted to the set $\Phi_0(\mathfrak{P}A)$ of all filters on subsets of $\mathfrak{P}A$, with the mapping $\mathfrak{P}$ assigning to each set its power set (for the terminology, see [2], p. 244 and 245)). (One has to be careful not to mix up $\mathfrak{P}\tau$ with the power set of the set τ.)

A relation R between topological spaces (E, σ), (F, ϱ) is said to be (σ, ϱ)-*continuous* (see [5]) if and only if

for each $(x, y) \in R$, *there is a* $(\sigma_{\mathscr{D}R}, \varrho)$-*continuous mapping* φ *on* $\mathscr{D}R$ *into* F *such that* $(x, y) \in \varphi \subseteqq R$.

Equivalently, *R is (σ, ϱ)-continuous if and only if R is the union of some set consisting of $(\sigma_{\mathscr{D}R}, \varrho)$-continuous mappings on $\mathscr{D}R$ into F.* ($\mathscr{D}R$ denotes the domain of R.)

Examples.

3. A mapping is continuous as a relation if and only if it is continuous in the usual sense. Especially, the identical mapping on A is (τ, τ)-continuous. Also, $A \times A$ is (τ, τ)-continuous.

4. Each congruence relation of each topological group is continuous. The same holds for topological rings.

5. The natural order relation on the real line (endowed with its usual topology) is continuous.

6. If (A, τ) is the Cartesian plane, then the relation R defined by "xRg if and only if $x \in g$ and g is a straight line" for all $(x, g) \in A \times \mathfrak{P}A$ is $(\tau, \mathfrak{P}\tau)$-continuous.

Proposition 2 ([3], "Satz 2"). *If Θ is a (τ, τ)-continuous equivalence relation on A, then the quotient topology τ/Θ and the trace $(\mathfrak{P}\tau)_{A/\Theta}$ of the topology $\mathfrak{P}\tau$ in the quotient set A/Θ coincide.*

3. In this section, we are going to combine Section 1 with Section 2, and we make free use of the agreements made there. Let $(\mathfrak{A}, \tau)$ be a topological partial algebra, i.e., let all partial operations f_γ of the given partial algebra $\mathfrak{A}$ be $((\tau^{n_\gamma})_{\mathscr{D}f_\gamma}, \tau)$-continuous mappings. ($\mathscr{D}f_\gamma$ denotes the domain of f_γ.)

Theorem. *Assume that Θ is a (τ, τ)-continuous congruence relation of $\mathfrak{A}$ and all domains $\mathscr{D}f_\gamma$ of the f_γ with $\gamma < \beta$ are τ^{n_γ}-open sets. Then, $({}^\Theta\mathfrak{A}, \mathfrak{P}\tau)$ is a super-compact* (see [2], p. 243) *topological algebra and $(\mathfrak{A}/\Theta, \tau/\Theta)$ is (a topological partial algebra, namely) the relative topological (partial) subalgebra of $({}^\Theta\mathfrak{A}, \mathfrak{P}\tau)$ with A/Θ as the base set,* i.e. $\mathfrak{A}/\Theta$ is the relative (partial) subalgebra of ${}^\Theta\mathfrak{A}$ with A/Θ as the base set, and $\tau/\Theta = (\mathfrak{P}\tau)_{A/\Theta}$.

Proof. Theorem 1 in [4], "Satz 3" in [2], Propositions 1 and 2.

Corollary 1. *If all domains $\mathcal{D}f_\gamma$ of the f_γ are open, then $(\mathfrak{P}\mathfrak{A}, \mathfrak{P}\tau)$ is a supercompact topological algebra and, up to an algebraic and topological isomorphism, $(\mathfrak{A}, \tau)$ is a relative topological (partial) subalgebra of $(\mathfrak{P}\mathfrak{A}, \mathfrak{P}\tau)$.*

Proof. Apply Theorem for $\Theta = id_A$ (see Examples 2 and 3) and use that $(\mathfrak{A}, \tau)$ is isomorphic to $(\mathfrak{A}/id_A, \tau/id_A)$.

Let E be the set $\{X \mid X \subseteqq A$ and card $X \leqq 1\}$ and define, for each $\gamma < \beta$, h_γ to be the restriction of the mapping ${}^{(id_A)}f_\gamma$ to the set E^{n_γ} ($= n_\gamma$-th Cartesian power of the set E). Then, $\mathfrak{E} = (E, (h_\gamma)_{\gamma<\beta})$ is an algebra; if $\mathfrak{A}$ is not an algebra itself, $\mathfrak{E}$ is the smallest subalgebra of $\mathfrak{P}\mathfrak{A}$ into which $\mathfrak{A}$ can be embedded isomorphically. (E has just one element more than A; cf. Grätzer [1], p. 80, proof of Theorem 1.) If we include in this statement the topological situation, we obtain

Corollary 2. *Assume that all domains $\mathcal{D}f_\gamma$ of the f_γ are open. Then: $(\mathfrak{E}, (\mathfrak{P}_\tau)_E)$ is a supercompact topological algebra, namely a topological subalgebra of $(\mathfrak{P}\mathfrak{A}, \mathfrak{P}\tau)$, and, up to an algebraic and topological isomorphism, $(\mathfrak{A}, \tau)$ is a relative topological (partial) subalgebra of $(\mathfrak{E}, (\mathfrak{P}\tau)_E)$.*

Of course, in the case that $\mathfrak{A}$ is not an algebra, $(\mathfrak{E}, (\mathfrak{P}\tau)_E)$ is the smallest topological subalgebra of $(\mathfrak{P}\mathfrak{A}, \mathfrak{P}\tau)$ which extends $(\mathfrak{A}, \tau)$ to a topological algebra (provided that all $\mathcal{D}f_\gamma$ are open).

References

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