HOMOTOPICAL STRUCTURE OF LINEAR GROUPS OF BANACH SPACES

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Some geometrical conditions are given for a Banach space X which implicate contractibility of the linear group GL(X), i.e., the group of all automorphisms of X with the topology induced by the norm

$$\|A\| = \sup \{\|Ax\| : \|x\| \leqq 1\}.$$

Definition 1. A Banach space X is *weakly infinitely decomposed* (WID) if

a) there exists a total system of disjoint projections $\{P_k, k \geqq 0\}$, i.e., $P_k P_i = P_i P_k = 0$, $\forall i \neq k$, and $(P_k x = 0, \forall k) \Rightarrow x = 0$;

b) all images $P_k X$, $k \geqq 0$, are isomorphic to X, more exactly, there exist isomorphisms $T_k : P_k X \xrightarrow{\sim} X$, $\forall k \geqq 0$;

c) X is isomorphic to its Cartesian square $X \times X$;

d) there exist bounded operators (left and right shifts) $S, S' : X \to X$ such that $T_k P_k S x = T_{k+1} P_{k+1} x$, $\forall k \geqq 0$ and $T_k P_k S' x = T_{k-1} P_{k-1} x$, $\forall k \geqq 1$, $P_0 S' x = 0$.

e) for any $B : X \to X$ there exists an operator $\tilde{B} : X \to X$ such that $T_k P_k \tilde{B} x = B T_k P_k x$, $\forall k \geqq 0$, $\forall x \in X$, i.e., the diagonal representation of $L(X)$ is continuous.

Definition 2. A Banach space X has the property of *smallness of operator blocks* (SOB) if for any compact family $B = (b)$ of operators in X and $\varepsilon > 0$ there exist projections Q_1 and Q_2 such that $Q_1 Q_2 = Q_2 Q_1 = 0$, its images $Q_i X$, $i = 1, 2$, are isomorphic to X, and $\|Q_1 b Q_2\| < \varepsilon$, $\forall b \in B$.

The pointing out of these conditions by the author of [1] is based on Kuiper's, 1965, and Neubauer's, 1967, constructions, and the further generalization of their results is the following

Theorem ([1], § 2). *Let a Banach space X have properties* WID *and* SOB. *Then* GL(X) *is contractible to* 1_X.

This statement has been used for a proof of contractibility of GL(X) for particular Banach spaces, namely, a) $L^p[0, 1]$, $1 < p < \infty$ (C. McCarthy and the author, [1], § 5); b) $C^k(M)$, $k \geqq 1$, M is a differentiable manifold ([1], § 4); c) $L^1[0, 1]$ (I. Edelstein, E. Semenov and the author [3], [1], § 4); d) $C(K)$ for a wide class of compacts (the same authors [3], Theorem 1).

Speaking in a more detailed way, the group $\mathrm{GL}(C(K))$ is contractible if K is one of the following compact Hausdorff spaces: 1) an uncountable compact metric space; 2) an infinite compact topological group; 3) an infinite product of non-one-point compact metric spaces; 4) the Stone space of an infinite homogeneous measure algebra; 5) βN — the Stone-Čech compactification of integers.

Nevertheless there exist ([2], § 9) such compacts K that $\mathrm{GL}(C(K))$ is not contractible. More precisely, let K_1 be a compact of ordinals less than or equal to ω_1, the first uncountable ordinal, with the interval topology; then $\mathrm{GL}(C_R(K_1)) \simeq Z_2$ and $\mathrm{GL}(C_c(K_1)) \simeq S^1$. More generally, if K_n = the union of n copies of K_1 then $\mathrm{GL}(C_R(K_n)) \simeq O(n)$ and $\mathrm{GL}(C_c(K_n)) \simeq U(n)$.

The James spaces also give homotopically-non-trivial linear groups. The above is a brief resume of [1], [2], [3].

References

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