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LOCAL BEHAVIOR IN SHAPE THEORY

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This is a report on some aspects of my joint work with G. Kozlowski which is to appear in a series of papers [8,9,10]. Although as originally conceived shape theory was designed to deal with global properties of metric compacta, we show that it is also related to the local properties of paracompacta. We prove that any LC^n paracompactum X of dimension $\le n$ is shape dominated by a polyhedron of dimension $\le n$. From this and other results it follows that such an X is an ANSR (absolute neighborhood shape retract), thus providing a shape version of the classical result that an LC^n compact metric space of dimension $\le n$ is an ANR. The importance of the classical result lay in the fact that it translated local homotopy property into information about extending maps. Our shape version likewise yields information about extending shape maps.

We also generalize movability to arbitrary topological spaces as an extension of uniform movability and obtain a shape invariant which is strictly stronger than movability for ANR-systems. We show that every LC^{n-1} paracompactum of dimension $\le n$ is uniformly movable. The strength of our definition of uniform movability is illustrated by the theorem: If (X,x) is a uniformly movable pointed continuum with trivial shape groups then (X,x) has trivial shape. This theorem is not true if one uses the weaker forms of movability. For LC^n paracompacta we also show that the shape groups and the homotopy groups are naturally isomorphic. As an application of this we have the theorem: For metric spaces a proper map $f:X \to Y$ of X onto Y such that $f^{-1}(y)$ is approximately k-connected $(0 \le k \le n)$ for all $y \in Y$ induces for each $x \in X$ an isomorphism of the nth shape group of (X,x) with that of $(Y,f(x))$.

1. Uniform Movability. In [1] Borsuk introduced the notion of movability for metric compacta. In [15] Mardešić and Segal generalized this notion to arbitrary compacta. Here we give a categorical description of this property which applies to arbitrary topological spaces and note it is a shape invariant. In fact, our version of movability is a generalization of uniform movability [16]. The reader is referred to [8] for a description of the natural

transformation approach to shape theory. This approach is essentially the same as that of Mardešić [13] except that he uses shape maps instead of natural transformations. One should note that a shape map from X to Y is a natural transformation from Π_Y to Π_X (i.e., $\mathrm{Mor}_{Sh}(X,Y) = \mathrm{Transf}(\Pi_Y, \Pi_X)$).

Definition 1. A space X is said to be uniformly movable provided, that for each map $f: X \to P$ of X into a (possibly infinite) polyhedron P, there exists a polyhedron Q and natural transformations $\Phi: \Pi_X \to \Pi_Q$, $\Psi: \Pi_Q \to \Pi_X$ such that $\Psi\Phi[f] = [f]$.

Remark. Since any natural transformation $\Psi: \Pi_Q \to \Pi_X$ satisfies $\Psi = g^{\#}$ for a map (unique up to homotopy) $g: X \to Q$, the condition of the above definition can be stated: for each map $f: X \to P$ there exist a polyhedron Q, maps $g: X \to Q$, $\phi: Q \to P$, and a natural transformation $\Phi: \Pi_X \to \Pi_Q$ such that $\phi g \simeq f$ and $\Phi[f] = [\phi]$.

Theorem 1. For any space uniform movability implies movability.

Theorem 2. If (X,x) is a uniformly movable pointed continuum with the shape groups $\underline{\pi}_n(X,x) = 0$, for all $n \geq 1$, then $\mathrm{Sh}(X,x) = 0$.

Corollary. The only uniformly movable compact connected abelian topological group with $\underline{\pi}_1(X) = 0$ is 0.

For an example which shows the converse of Theorem 1 is false and Theorem 2 is false if one uses movability instead of uniform movability the reader is referred to [9].

2. ANSE's and ANSR's. In this section we generalize the notion of extensor to the theory of shape for paracompacta. We also generalize the notions of FANR [2] and ANSR [14] to paracompacta. The starting point is the generalization of the neighborhood extension of maps to the neighborhood extension of shape morphisms. The universal quantification of this property gives the concept of absolute neighborhood shape extensor (ANSE).

Definition 2. We say Y is an absolute neighborhood shape extensor for paracompacta (ANSE) if for any natural transformation $\Phi:\Pi_Y \to \Pi_A$, where A is any closed subset of an arbitrary paracompactum X, there is a closed neighborhood N of A and a natural transformation $\Psi:\Pi_Y \to \Pi_N$ such that $\rho\Psi = \Phi$ (where $\rho:\Pi_N \to \Pi_A$ denotes the restriction). In the ANR-systems approach this implies that any compactum Y is an absolute neighborhood shape extensor if any shape map $\underline{f}:\underline{A} \to \underline{Y}$ can be extended to a shape map $\underline{F}$ of a closed neighborhood N of A in X. (Here $\underline{F}$ extends $\underline{f}$ means $\underline{F}\ \underline{i} \simeq \underline{f}$ where $\underline{i}$ is a shape map of A into N induced by the inclusion $i:A \to N$.)

We will also generalize the following description of absolute neighborhood retracts for compacta in shape theory due to Mardešić [14] to paracompacta. Mardešić's definition was a generalization of Borsuk's [2] fundamental absolute neighborhood retracts (FANR's) to the compact Hausdorff case. Mardešić says that a compactum Y is a absolute neighborhood shape retract provided, for every compactum Z, $Y \subset Z$, there exists a closed neighborhood N of Y in Z, such that Y is a shape retract of N (i.e., there is a shape map $\underline{r}: \underline{N} \to \underline{Y}$ such that $\underline{r}\ \underline{i} \simeq \underline{1}_Y$, where $i: Y \to N$ is the inclusion map).

Definition 3. The paracompactum Y is said to be an absolute neighborhood shape retract (ANSR) if, whenever Y is a closed subset of a paracompactum Z, there exist a neighborhood N of Y in Z and a natural transformation $\Psi: \Pi_Y \to \Pi_N$ such that $\rho\Psi = 1_{\Pi_Y}$ (where $\rho: \Pi_N \to \Pi_Y$ is the restriction). (Every compact ANSR is an ANSR (in the sense of Mardešić) since for any natural transformation $\Phi:\Pi_Y \to \Pi_X$ there exists a map of systems $\underline{f}: \underline{X} \to \underline{Y}$ such that $\underline{f}^{\#} = \Phi$.)

Theorem 3. If a paracompactum Y is an ANSE then it is an ANSR.

Theorem 4. A compactum Y is an ANSR (in the sense of Mardešić) iff it is an ANSE.

Theorem 5. Any space shape dominated by an ANSE is also an ANSE.

The following is a restatement in shape theory of a result of [11].

Theorem 6. Any polyhedron P is an ANSE.

3. Locally Well-behaved Paracompacta. In this section we describe how shape theory can be used effectively to deal with some local homotopy properties of paracompacta. Making use of the techniques of partial realizations we are able to obtain the following results.

Theorem 7. Any LC^n paracompactum X of (covering) dimension $\leq n$ is shape dominated by some polyhedron of dimension $\leq n$.

Since any polyhedron with the metric topology is an ANR we have

Corollary. Any LC^n paracompactum X of dimension $\leq n$ is shape dominated by an ANR.

Theorem 8. Any LC^n paracompactum X of dimension $\leq n$ is an ANSE and therefore an ANSR.

Remark. Since an ANSR may behave badly locally there is no chance of extending the compact metric result, ANR $\rightarrow LC^n$, to paracompacta. On the other hand, an example due C.W. Saalfrank shows that the compact metric result, at most n-dimensional and $LC^n \rightarrow$ ANR, cannot be extended to compact Hausdorff spaces. However, Theorem 8 does extend it to paracompacta in shape theory, i.e., at most n-dimensional and $LC^n \rightarrow$ ANSR.

In the next theorem uniformly n-movable is a stratification of uniformly movable (see [8]).

Theorem 9. Every LC^{n-1} paracompactum is uniformly n-movable. (This extends Borsuk's result [4] in the compact metric case.)

Corollary. Every LC^{n-1} paracompactum of dimension $\leq n$ is uniformly movable. (In the compact metric case this was first obtained by Mardešić [12] and in [17].)

4. Summary. We now summarize in diagram form the previous results and classical results on locally well-behaved compacta. An arrow ($\rightarrow$) indicates class inclusion and a broken arrow (-n$\rightarrow$) indicates class inclusion under the additional hypothesis that the dimension of the space in question is $\leq n$. Here SDP indicates a space dominated by a polyhedron.

Classically we have for metric spaces:

(I) $$LC^n \underset{-n\rightarrow}{\longleftarrow} ANE \leftrightarrow ANR$$

and for compacta:

(II) $$LC^n \leftarrow ANE \leftrightarrow ANR$$

In shape theory we have for compacta:

(III) $$\begin{array}{ccccccc} LC^n & \xrightarrow{\quad n \quad} & SDP & \leftrightarrow & ANSE & \leftrightarrow & ANSR \\ \downarrow & & \downarrow & & & & \\ LC^{n-1} & & \downarrow & & & & \\ \downarrow & & \downarrow & & & & \\ \text{uniformly } n\text{-movable} & \underset{-n\rightarrow}{\longleftarrow} & \text{uniformly movable} & & & & \end{array}$$

and for paracompacta

(IV) $$\begin{array}{ccccccc} LC^n & \xrightarrow{\quad n \quad} & SDP & \longrightarrow & ANSE & \longrightarrow & ANSR \\ \downarrow & & \downarrow & & & & \\ LC^{n-1} & & \downarrow & & & & \\ \downarrow & & \downarrow & & & & \\ n\text{-movable} & \underset{-n\rightarrow}{\longleftarrow} & \text{uniformly movable} & & & & \end{array}$$

We do not know if ANSR $\rightarrow$ ANSE or if ANSE $\rightarrow$ SDP for paracompacta. The role of SDP has been investigated by Edwards and Geoghegan in [5] and [6], and Geoghegan and Lacher in [7].

5. <u>Shape Groups and the Vietoris Theorem</u>. In [10] we define the <u>nth shape group</u> of a topological space X at x (denoted by $\underline{\pi}_n(X,x)$) as the collection of natural transformations from Π_X to Π_{S^n} with addition defined as follows:

$$(\Phi + \Psi)[f] = \Phi[f] + \Psi[f]$$

where $f\colon X \rightarrow P$, a polyhedron, the last addition is in $\Pi_{S^n}(P)$ and we are working in the pointed case. The relation with the homotopy groups $\pi_n(X,x)$ given by $[\phi] \rightarrow \phi^{\#}$ for a map $\phi\colon S^n \rightarrow X$ is a homomorphism.

Using partial realization techniques we prove the following result.

<u>Theorem 10</u>. For LC^n paracompacta $\pi_n(X)$ is naturally isomorphic to $\underline{\pi}_n(X)$.

As an application of this result we prove a shape version of the Vietoris (-Smale) Theorem. S. Bogatyĭ and K. Kuperberg obtained such a result in the compact metric case. Our result overlaps with the recent work of J. Dydak [4].

Theorem 11. For metric spaces and a perfect map $f: X \to Y$ such that $f^{-1}(y)$ is approximately k-connected, $0 \leq k \leq n$, for all y in Y the induced homomorphism

$$f_{\#}: \underline{\pi}_n(X) \to \underline{\pi}_n(Y)$$

is an isomorphism.

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