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RAMSEY TOPOLOGICAL SPACES

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In this communication we are interested in the following problems:

<u>Problem 1</u>: Given a topological space $(X,\mathcal{T})$ and a cardinal γ does there exist a topological space $(Y,\mathcal{U})$ such that for every mapping $c: Y \longrightarrow \gamma$ there exists a topological embedding $f: (X,\mathcal{T}) \longrightarrow (Y,\mathcal{U})$ such that $c \circ f$ is a constant mapping.

<u>Problem 2</u>: Given a topological space $(X,\mathcal{T})$ and a well ordering $\leq$ of X does there exist a topological space $(Y,\mathcal{U})$ such that for every well ordering $\preccurlyeq$ of Y there exists a topological embedding $f: (X,\mathcal{T}) \longrightarrow (Y,\mathcal{U})$ which is also a monotonne mapping $(X,\leq) \longrightarrow (Y,\preccurlyeq)$.

Both these questions belong to the generalized partition theory. In this note we sketch a background of these problems and then we state partial solutions and problems related to the above questions.

Generalized partition theory is motivated by the following theorem of Ramsey [4] :

For each choice of positive integers k, α, γ there exists a positive integer β with the following property: for every mapping $c: [\beta]^k = \{ b \subseteq \beta \ ; \ |b| = k \} \longrightarrow \gamma$ there exists a subset $\alpha' \in [\beta]^{\alpha}$ such that the mapping c restricted to the set $[\alpha']^k$ is a constant. (The mapping c is usually called a colouring.)

This is one of the most applied combinatorial facts. Ramsey theorem may be used for counterexamples: the theorem tells that one cannot distribute k-element subsets of large sets in such a way that no homogeneous subset of a given size occurs.

Ramsey theorem was generalized in many ways. The stronger Ramsey type theorem we have the better examples we may construct. [1] , [2] and [3] are recent surveys of various generalizations of Ramsey theorem.

Here we are interested in Ramsey type questions related directly to topological spaces. For this is best to formulate Ramsey theorem in categorical terms. This may be done as follows:

Let $\mathcal{K}$ be a category, A, B its objects, γ a cardinal number. We say that an object C of $\mathcal{K}$ is <u>(A,γ)-Ramsey for B</u> if for every mapping $c : \mathcal{K}(A,C) \longrightarrow \gamma$ there exists a morphism $f \in \mathcal{K}(B,C)$ and an $\iota \in \gamma$ such that $c(f \circ g) = \iota$ for all $g \in \mathcal{K}(A,B)$.
($\mathcal{K}(A,B)$ is the set of all morphisms from A to B .)

This statement is denoted by $B \xrightarrow[\gamma]{A} C$.

Usually, categories are considered with monomorphisms only. Then the symbol $B \xrightarrow[\gamma]{A} C$ has the intuitive meaning of a "combinatorially strenghtened embedding" .

As an extremal case we say that the category $\mathcal{K}$ has <u>A-partition property</u> if for every $B \in \mathcal{K}$ and γ there exists $C \in \mathcal{K}$ such that $B \xrightarrow[\gamma]{A} C$; $\mathcal{K}$ is said to be <u>Ramsey</u> if $\mathcal{K}$ has A-partition property for every $A \in \mathcal{K}$.

Ramsey theorem states that in the finite set theory the category of all sets and all 1-1 mappings is Ramsey.

It is easy to see that the categories of all topological spaces, metric spaces, compact spaces and other most frequent topological categories fail to be Ramsey. The reason for this is the fact that $X \xrightarrow[2]{X} Y$ for no topological space Y providing that the space X contains a proper subspace isomorphic to X . (A proof is simple: Let $\binom{Y}{X}$ be the set of all topological subspaces of Y which are isomorphic to X. Consider the set of all partial mappings $c : \binom{Y}{X} \longrightarrow \{0,1\}$ which satisfy the following condition: if $X' \in \binom{Y}{X}$ and $c(X')$ is defined then there exists $X'' \in \binom{X'}{X}$ such that $X'' \neq X'$ and $c(X'') \neq c(X')$. Using Zorn's lemma there exists a maximal - with respect to the inclusion - partial mapping $c: \binom{Y}{X} \longrightarrow$ $\longrightarrow \{0,1\}$ with the above property. One may check easily that c is defined on $\binom{Y}{X}$. c violates the definition of $X \xrightarrow[2]{X} Y$.)

However for certain topological spaces the situation is more promising. The following holds:

Theorem 1: The class of all topological spaces has 1-point partition property.
Explicitely: for every topological space X and a cardinal γ there exists a topological space Y such that for every mapping $c: Y = \binom{Y}{1} \longrightarrow \gamma$ there exists a topological embedding $f: X \longrightarrow Y$ such that $c \circ f$ is a constant mapping.

This answers the Problem 1. However we are unable to answer the same question for the class of all Hausdorff topological spaces and spaces with higher separation axioms . Perhaps the most natural question is the following:
Unit interval problem: Is it true that $I \xrightarrow[2]{1} I^{\alpha}$ for a cardinal α ? (I is the closed unit interval, 1

denotes the 1-point topological space.)

Proof of Theorem 1 is provided by the following example: Consider the set $Y = X^{\gamma}$. For $x^0 = (x^0_\iota \,;\, \iota < \gamma) \in Y$, $\beta < \gamma$, and a neighborhood U_β of x^0_β define the set $U(x^0, U_\beta)$ by $(x_\iota \,;\, \iota < \gamma) \in U(x^0, U_\beta)$ iff $x_\iota = x^0_\iota$ for $\iota < \beta$, $x^0_\beta \neq x_\beta \in U_\beta$, and $x_\iota \in X$ for $\iota > \beta$. Let the set Y be endowed with the topology given by the subbase neighborhoods of the form $\{x^0\} \cup \bigcup (U(x^0, U_\beta) \,;\, \beta < \gamma)$. It is possible to prove $X \xrightarrow[\gamma]{1} Y$. (The details are going to appear elsewhere.)

Concerning Problem 2 we do not have a similarily general result. Using graphs we can prove only:

Theorem 2: For every finite metric space (X, ϱ) there exists a finite metric space (Y, σ) such that for every orderings $(X, \leq)$ and $(Y, \preccurlyeq)$ there exists a mapping $f : X \longrightarrow Y$ which is both a monotonne mapping $(X, \leq) \longrightarrow (Y, \sigma)$ and an embedding $(X, \varrho) \longrightarrow (Y, \sigma)$.

This theorem leads to the hardest result of finite partition theory (ofcourse finite metric spaces may be identified with graphs), see [2].

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