T_3-COMPLETIONS OF CONVERGENCE VECTOR SPACES

Bernd Müller

Mannheim

I. Introduction

The aim of this paper is to construct, for a T_3-convergence vector space E (abbreviated by T_3-cvs), a T_3-completion $\hat{E}$ with the following properties: $\hat{E}$ is a complete T_3-cvs, possesses the usual universal property within the category of T_3-cvs and contains a subspace isomorphic to E. First we give an example of a T_3-cvs F for which there exists no complete convergence vector space containing F as a subspace. If we consider a completion C(F) of F in the category of uniform convergence spaces or Cauchy spaces which contains F as a subspace (see e.g. [9], [1o]), this example shows that C(F) cannot be a convergence vector space. Therefore we characterize those T_3-cvs E which possess a T_3-completion $\hat{E}$. For example, every subspace E of a complete T_3-cvs M possesses a T_3-completion $\hat{E}$ but, in general, $\hat{E}$ is not isomorphic to a subspace of M, even if E is dense in M (see example III.3). Other examples of T_3-cvs possessing a T_3-completion are locally precompact T_3-cvs (see [8]). In this case, the T_3-completion is a locally compact T_3-cvs. Finally, it is mentioned without proof, that for certain vector sublattices A of $C_c(X)$, the algebra of all continuous real valued functions on the convergence space X, endowed with the continuous convergence structure (see [1]), the T_3-completion is isomorphic to the inductive limit of the family $\{ a^n(A) : n \in \mathbb{N} \}$, taken in the category of all convergence spaces.

A convergence space will be always a convergence space in the sense of H.R.Fischer (see [1]). An $\mathbb{R}$-vector space E endowed with a convergence structure λ is called a convergence vector space (cvs) iff the algebraic operations are continuous. This can be described internally: A convergence structure λ on an $\mathbb{R}$-vector space E is a convergence vector space structure iff $\forall\ x \in E$ the family λx of all filters converging to x has the following properties:

1. $\Phi + \Psi \in \lambda o \quad \forall\ \Phi,\Psi \in \lambda o$
2. $\alpha\Phi \in \lambda o \quad \forall\ \alpha \in \mathbb{R}\ ,\ \forall\ \Phi \in \lambda o$
3. $\mathbb{V}\Phi \in \lambda o \quad \forall\ \Phi \in \lambda o$, where $\mathbb{V}$ is the neighborhood filter of o in $\mathbb{R}$
4. $\mathbb{V}x \in \lambda o \quad \forall\ x \in E$
5. $\lambda x = x + \lambda o \quad \forall\ x \in E$

For a subset U of a cvs E we define the adherence a(U) to be the set $\{\ y : y \in E,\ \exists$ a filter Φ converging to y with $U \in \Phi\ \}$ and $a^{n+1}(U) := a(a^{n}(U)) \quad \forall\ n \in \mathbb{N}$. U is called dense in E if a(U)=E, and closed if a(U)=U. The closure of U is the smallest closed subset of E containing U. A cvs E is called regular if for all $x \in E$ and for all filters Φ converging to x, the filter $a(\Phi)$ generated by $\{\ a(U): U\in\Phi\}$ also converges to x. A T_3-cvs is a separated regular cvs. We denote by $L(E,F)$ the set of all continuous linear mappings of a cvs E into a cvs F. A mapping $T \in L(E,F)$ is called an isomorphism from E into F if T is injective and $T^{-1}: T(E) \to E$ is continuous. The cvs E and F are called isomorphic if there exists an isomorphism from E onto F. All cvs considered in this paper are vector spaces over $\mathbb{R}$.

II. Construction of a T_3-completion

Let us begin with the usual definition of a Cauchy filter.

Definition II.1: *A filter* Θ *in a cvs* E *is called a Cauchy filter if* $\Theta - \Theta$ *converges to* o *in* E. *A Cauchy filter* Θ *in* E *is called bounded if* $\mathbb{V}\Theta$ *converges to* o *in* E *where* $\mathbb{V}$ *is the neighborhood filter of* o *in* $\mathbb{R}$. *A cvs* E *is called complete if every Cauchy filter in* E *converges.*

Since every Cauchy filter of a complete cvs F is bounded, it is a necessary condition for a cvs E to be a subspace of a complete cvs that every Cauchy filter is bounded. We now give an example of a T_3-cvs E for which not every Cauchy filter is bounded.

Example II.2: Consider $E := \bigoplus_{i\in\mathbb{N}} R_i$ with $R_i = \mathbb{R}$ $\forall\, i \in \mathbb{N}$, where the direct sum is taken in the category of all cvs. $\forall\, m,n \in \mathbb{N}$ define

$F_{m,n} := \{ (x_j)_{j\in\mathbb{N}} : (x_j)_{j\in\mathbb{N}} \in E,\ x_1=\ldots=x_n=o,\ |x_j| \leq \frac{1}{m}\ \forall\, j \in \mathbb{N} \}$

Let F be the filter generated by $\{ F_{m,n} : m,n \in \mathbb{N} \}$, and let λ be the convergence structure on E defined in the following way:

A filter Φ converges to x in $(E,\lambda) \iff x - \Phi - F$ converges to o in E.

It is not hard to see that (E,λ) is a T_3-cvs for which every bounded Cauchy filter converges. But the sequence $(x_r)_{r\in\mathbb{N}}$, defined by

$x_r = (x_{r,j})_{j\in\mathbb{N}}$ with $x_{r,j} := \begin{cases} \frac{1}{j} & \text{if } j \leq r \\ o & \text{if } j > r \end{cases}$, is an unbounded Cauchy sequence in (E,λ).

Definition II.3: *A complete* T_3*-cvs* $\hat{E}$ *is called a* <u>T_3*-completion*</u> *of a* T_3*-cvs* E *if the following holds:*

1. *There exists an isomorphism* i *from* E *into* $\hat{E}$, *such that* $\hat{E}$ *is the closure of* i(E).
2. $\forall$ *complete* T_3*-cvs* M *and* $\forall\, T \in L(E,M)$ $\exists\, \hat{T} \in L(\hat{E},M)$ *such that* $T = \hat{T}\circ i$.

Remark: A T_3-completion of a T_3-cvs E is uniquely determined if it exists, and for every separated topological vector space F, the usual topological separated completion of F is also the completion of F in the category of all T_3-cvs.

Every subspace E of a complete T_3-cvs F has the following property:

(*) A filter Φ converges to o in E $\iff$ $\forall$ complete T_3-cvs M and $\forall\, T \in L(E,M)$, the filter $T(\Phi)$ converges to o in M.

Since the property (*) is a necessary condition for E to have a T_3-completion, we define:

Definition II.4: *A separated cvs* E *is called* *a-regular* *if it has property* (*).

Remark: Every a-regular cvs E is a T_3-cvs and every Cauchy filter in an a-regular cvs is bounded. As example II.2 shows, there are T_3-cvs being not a-regular.

Let E be an a-regular cvs. We now show that E possesses a T_3-completion. For this purpose let C be the set of all Cauchy filters in E. On C we define a relation $\sim$ by

$$\Phi \sim \Psi \iff \text{the filter } \Phi - \Psi \text{ converges to o in E.}$$

Since for all $\alpha,\beta \in \mathbb{R}$ and for all $\Phi,\Psi \in C$ the filter $\alpha\Phi + \beta\Psi$ is a Cauchy filter in E, the quotient $E_c := C/_\sim$ carries a vector space structure in a natural way. Define a linear mapping $i : E \to E_c$ in the following way: $i(x) := \mu(\dot{x})\ \forall\ x \in E$, where $\mu : C \to C/_\sim$ is the quotient mapping and $\dot{x}$ is the filter generated by $\{x\}$. Since E is a separated cvs, this mapping i is injective. We now want to construct a convergence vector space structure on E_c such that the mapping $i : E \to E_c$ is continuous. For every subset $U \subseteq E$ let us denote by U_c the set $\{ \mu(\Psi) : \Psi \in C, U \in \Psi \} \subseteq E_c$. Let H be a filter on E_c. We define:

H converges to $\mu(\Psi)$ in $E_c \iff \exists$ a filter Θ converging to o in E such the filter Θ_c generated by $\{ U_c : U \in \Theta \}$ is coarser than $\mu(\Psi)-H$.

Due to this definition, E_c is a convergence vector space. Since E is a T_3-cvs, E_c is a separated cvs and the mapping $i : E \to E_c$ is an isomorphism from E into E_c. For every $\Phi \in C$, the filter $i(\Phi)-\mu(\Phi)$ is finer than the filter $(\Phi-\Phi)_c$ generated by $\{ (F-F)_c : F \in \Phi \}$, which implies that $i(\Phi)$ converges to $\mu(\Phi)$ in E_c.

Propsition II.5: *For every a-regular cvs* E *the cvs* E_c *has the following properties:*

a. *There exists an isomorphism* i *from* E *into* E_c, *such that* i(E) *is dense in* E_c.

b. *For every Cauchy filter* Φ *in* E *the filter* $i(\Phi)$ *converges in* E_c.

c. *If* E *is a separated topological vector space,* E_c *is the usual separated topological completion of* E .

d. $\forall$ *complete* T_3-*cvs* M *and* $\forall\ T \in L(E,M)\ \exists\ T_c \in L(E_c,M)$ *such that* $T = T_c \circ i$.

Proof: We will only prove property d. Let M be a complete T_3-cvs and $T \in L(E,M)$. For every $x \in E_c$ we define $T_c(x)$ to be the limit of the filter $T(\Phi)$ in M, where Φ is any Cauchy filter in E with $x = \mu(\Phi)$. For all subsets $U \subseteq E$ the subset U_c of E_c has the following property: $y \in U_c \Leftrightarrow \exists\ \Phi \in C$ with $U \in \Phi$ and $y = \mu(\Phi)$.
This implies $T(U_c) \subseteq a(T(U))$, and therefore T_c is continuous.

Proposition II.6: *For every a-regular cvs* E *there exists an a-regular cvs* A(E) *with the following properties:*

a. *There exists an isomorphism* i *from* E *into* A(E) , *such that* i(E) *is dense in* A(E).

b. *For every Cauchy filter* Φ *in* E *the filter* $i(\Phi)$ *converges in* A(E).

c. *If* E *is a separated topological vector space,* A(E) *is the separated topological completion of* E.

d. *For every complete* T_3-*cvs* M *and for every* $T \in L(E,M)$ *there exists an operator* $A(T) \in L(A(E),M)$ *with* $T = A(T) \circ i$.

Proof: Let M be the category of all complete T_3-cvs and $|M|$ the class of all objects of M . Let E_c be the cvs constructed in proposition II.5 . For all $M \in |M|$ and for all $T \in L(E_c,M)$ let us denote by $\lambda_{M,T}$ the coarsest convergence vector space structure on E_c for which T is continuous. Since $\lambda_{M,T}$ is coarser than the convergence structure of E_c, there exists a coarsest convergence vector space structure λ which is finer than $\lambda_{M,T}$ for all $M \in |M|$ and all $T \in L(E_c,M)$. Let us denote by $A(E)$ the vector space E_c endowed with this convergence structure λ. Since for every $o \neq x \in A(E)$ there exists an $M \in |M|$ and $T \in L(A(E),M) = L(E_c,M)$ with $T(x) \neq o$, it is easy to see that $A(E)$ is a-regular. Let us now prove that the mapping $i : E \to E_c$ is also an isomorphism from E into $A(E)$. For this purpose let Φ be a filter in E, such that $i(\Phi)$ converges to o in $A(E)$. For every $M \in |M|$ and $T \in L(E,M)$ there exists a map $T_c \in L(E_c,M)$ with $T = T_c \circ i$. Since T_c is also a continuous mapping from $A(E)$ into M, the filter $T_c(i(\Phi))$ converges to o in M. From $T = T_c \circ i$ it follows that $T(\Phi)$ converges to o in M. Since E is a-regular, Φ converges to o in E. The other properties, described in proposition II.6, follow from the corresponding properties of E_c in proposition II.5.

<u>Theorem II.7</u>: *A T_3-cvs E possesses a T_3-completion if and only if E is a-regular.*

<u>Proof</u>: Let E be an a-regular cvs. We define $E_1 := E$, $E_{n+1} := A(E_n)$ and we consider E_n as a subspace of E_{n+1} for all $n \in \mathbb{N}$. The inductive limit $\hat{E}$ of the family $\{ E_n : n \in \mathbb{N} \}$, taken in the category of all cvs, is a separated and complete cvs. To show that $\hat{E}$ is a regular cvs, we consider a filter Φ converging to o in $\hat{E}$. By definition of $\hat{E}$, there exists an $m \in \mathbb{N}$ and a filter Ψ in E_m, converging to o in E_m, such that the filter generated by Ψ in $\hat{E}$ is coarser than Φ. Take $V \in \Psi$ and $x \in a(V)$, the adherence of V built in $\hat{E}$. There exists a filter Θ with $V \in \Theta$ which converges to x in $\hat{E}$. One can find an $r \in \mathbb{N}$, $r \geq m$, such that $x \in E_r$, $E_r \in \Theta$ and $\Theta_r := \{ U \cap E_r : U \in \Theta \}$ is a filter in E_r which converges to x in E_r. Since E_m is a subspace of E_r, the filter $\Theta_m := \{ W \cap E_m : W \in \Theta \}$ is a Cauchy filter in E_m, and since every Cauchy filter of E_m converges in E_{m+1}, x is an element of E_{m+1}. Therefore we have $a(V) = a_{m+1}(V)$, where $a_{m+1}(V)$ is the ad-

herence of V taken in E_{m+1}. This implies that the filter $a(\Phi)$ generated by $\{ a(V) : V \in \Phi \}$ has a basis in E_{m+1} and converges to o in E_{m+1}, since E_{m+1} is regular. Therefore $\hat{E}$ is a complete T_3-cvs which contains E as a subspace, because E is a subspace of E_n for all $n \in \mathbb{N}$. E_n is a dense subspace of E_{n+1} $\forall n \in \mathbb{N}$, which implies that $\hat{E}$ is the closure of E. Now let M be a complete T_3-cvs and $T \in L(E,M)$. We define $T_1 := T$ and $T_{n+1} := A(T_n)$ $\forall n \in \mathbb{N}$. If we put $\hat{T}(x) := T_n(x)$ if x lies in E_n, we get a continuous linear mapping $\hat{T} : \hat{E} \to M$ with $\hat{T}(x) = T(x)$ $\forall x \in E$.

In the definition of a T_3-completion, a very strong property was required, namely the existence of an isomorphism i from the cvs E into its T_3-completion $\hat{E}$. If one is only interested in the existence of a continuous linear mapping $i : E \to \hat{E}$, one can show that every cvs E possesses a "T_3-completion". In the language of category theory, this can be formulated in the following way:

Proposition II.8: *There exists an epireflector* V *from the category* $\mathcal{L}$ *of all cvs into the category* M *of all complete* T_3*-cvs.*

Proof: Let us denote by $|M|$ the class of all objects of M and let E be a cvs. Let G be the vector space E, endowed with the coarsest convergence vector space structure for which $T : G \to M$ is continuous $\forall M \in |M|$ and $\forall T \in L(E,M)$. Since $H := \bigcap \{ T^{-1}(o) : M \in |M|, T \in L(E,M)\}$ is a closed subspace of G, the quotient $F := G/_H$ is an a-regular cvs. We define V(E) to be the T_3-completion of F. Let μ_E be the natural mapping from E into V(E). For every cvs F and $T \in L(E,F)$ let us define $V(T) \in L(V(E),V(F))$ to be the uniquely determined mapping from V(E) into V(F) with $V(T) \circ \mu_E = \mu_F \circ T$. Now it is not hard to see that V is an epireflector from $\mathcal{L}$ into M.

III. Vector sublattices of $C_c(X)$

In this section we will describe the T_3-completion of a vector sublattice of $C_c(X)$, the algebra of all continuous real valued functions on a convergence space X endowed with the continuous convergence structure (see [1]). For any subset A of $C_c(X)$ let us denote by $c_A X$ the set X carrying the coarsest convergence structure such that the mapping $i : X \to C_c(A)$, defined by $[i(x)](f) := f(x)$ $\forall\, x \in X$ and $\forall\, f \in A$, is continuous. It is easy to see that A is not only a subspace of $C_c(X)$, but also a subspace of $C_c(c_A X)$.

Proposition III.1: *Let* B *be a vector sublattice of* $C_c(X)$*, which separates points in* X *and contains the constant functions. Then the cvs* B_c *and* A(B)*, constructed in section II, are isomorphic to the adherence* a(B) *of* B*, taken in* $C_c(c_B X)$*.*

This proposition implies the following result:

Theorem III.2: *Let* X *be a convergence space and let* A *be a vector sublattice of* $C_c(X)$*, which separates points in* X *and contains the constant functions. Then* A *is also a subspace of* $C_c(c_A X)$ *and the inductive limit* $\hat{A}$ *of the family* $\{ a^n(A) : n \in \mathbb{N} \}$*, taken in the category of all cvs, is the* T_3*-completion of* A*, where* $\forall\, n \in \mathbb{N}$ *the spaces* $a^n(A)$ *are built in* $C_c(c_A X)$*.*

From Stone-Weierstraß theorems which can be found in [1] and [3] it follows:

Corollary: *Let* A *be a vector sublattice of* $C_c(X)$ *which separates points in* X *and contains the constant functions. Assume that* X *is a topological Lindelöf space with* $X = c_A X$ *or that* X *carries the coarsest topology, such that every* $f \in A$ *is continuous. Then* $C_c(X)$ *is the* T_3*-completion of* A*.*

Remark: There exists a topological space X and a vector sublattice A of $C_c(X)$, such that $a^{n+1}(A)\setminus a^n(A) \neq \emptyset \ \forall n \in \mathbb{N}$. This shows that in general an a-regular cvs is not dense in its T_3-completion.

Now we will construct a topological space X and a dense vector sublattice A of $C_c(X)$ such that $C_c(X)$ is not the T_3-completion of A.

Example III.3: Let us denote by $[o,\omega]$, resp. $[o,\Omega]$, the set of all ordinals less than or equal to the first countable, resp. first uncountable, endowed with the interval topology. In $[o,\Omega]$ we define a sequence by $x_1 := 1$ and $x_{n+1} := \lim\limits_{r\to\infty} rx_n \ \forall n \in \mathbb{N}$.
Define $T_1 := \{[o,\Omega] \times [o,\omega] \setminus \{(\Omega,\omega)\}\} \times \{1\}$ and $T_n := \{[o,\Omega] \times [o,\Omega] \setminus \{(\Omega,\Omega)\}\} \times \{n\} \quad \forall 1 < n \in \mathbb{N}$. In the topological sum $T := \sum\limits_{n\in\mathbb{N}} T_n$ identify $(x,\omega,1)$ with $(x,\Omega,2) \ \forall x\in[o,\Omega]\setminus\{\Omega\}$, $(\Omega,y,2n)$ with $(\Omega,y,2n+1)$ and $(z,\Omega,2n+1)$ with $(z,\Omega,2n+2)$ $\forall y,z \in [o,\Omega]\setminus\{\Omega\}$ and $\forall n \in \mathbb{N}$. Let Q be the quotient which arises from T by this identification, and let ψ be the quotient mapping. On $P := Q \cup \{a\}$, where $a \notin Q$, we define a topology in the following way: For every $x \in Q$ let U(x) be a basis of the neighborhood filter of x in P, where U(x) is the neighborhood filter of x in Q, and for $a \in P\setminus Q$ let $\{\bigcup\limits_{\nu\geq n} \psi(T_\nu) \cup \{a\} : n \in \mathbb{N}\}$ be a basis of the neighborhood filter of a. It is easy to see that P is a c-embedded topological space (see [4]). Define $y_m := \psi((x_m,x_m,m))$, $z_m := \psi((\Omega,m,1))$ $\forall m \in \mathbb{N}$ and $X := P\setminus\{y_m : m \in \mathbb{N}\}$. As a subspace of a c-embedded topological space, X is c-embedded. Now consider

$$A := \{ f : f \in C(X),\ f(z_{m+1}) = \lim_{k\to\infty} f(\psi((kx_m,kx_m,m+1))) \ \forall m \in \mathbb{N} \}$$

It is not hard to see that A is a point separating vector sublattice of C(X), which contains the constant functions. For every subset U of X let us denote by $\overline{U}^A$ the closure of U in the coarsest topology on X for which all $f \in A$ are continuous. Now for all $p \in X$ and all filters Θ converging to p in X, the filter $\overline{\Theta}$ generated by

$\{ \bar{U}^A : U \in \Theta \}$ converges to p in $c_A X$. This implies that the sequence $(z_m)_{m\in\mathbb{N}}$ convergies to a in $c_A X$. The linear mapping $\zeta : C_c(c_A X) \to \mathbb{R}$, defined by $\zeta(f) := \sum_{n\in\mathbb{N}} (\frac{1}{2})^n f(z_n) \quad \forall f \in C(c_A X)$, is continuous. Since A is a subspace of $C_c(c_A X)$, the restriction δ of ζ to A is continuous. The set $\{ z_m : m \in \mathbb{N} \}$ is not relatively compact in X, therefore δ has no continuous extension from A to $C_c(X)$. Finally it is not hard to see that A is dense in $C_c(c_A X)$ and in $C_c(X)$.

References

[1] E.Binz: Continuous Convergence on C(X), Lecture Notes in Mathematics, 469, Springer-Verlag (1975)

[2] S.Bjon: Vervollständigung streng ausgeglichener Limesvektorräume, to appear

[3] H.-P.Butzmann: Der Satz von Stone-Weierstraß in $C_c(X)$ und seine Anwendung auf die Darstellungstheorie von Limesalgebren, Habilitationsschrift, Univ. Mannheim (1974)

[4] H.-P.Butzmann,B.Müller: Topological, c-embedded spaces, General Topology and its Appl. 6, 17-2o (1976)

[5] S.Gähler,W.Gähler,G.Kneis: Vervollständigung pseudotopologischer Vektorräume, to appear in Math.Nachr.

[6] R.J.Gazik,D.C.Kent: Regular completions of Cauchy spaces via function algebras, Bull.Austr.Math.Soc. 11, 77-88 (1974)

[7] B.Müller: Vervollständigungen von Limesvektorräumen, Habilitationsschrift,Univ. Mannheim (1975)

[8] B.Müller: Locally Precompact Convergence Vector Spaces, Conference proceedings, Univ.of Nevada,Reno, 155-172 (1976)

[9] E.E.Reed: Completions of uniform convergence spaces, Math.Ann. 194, 83-1o8 (1971)

[1o] O.Wyler: Ein Komplettierungsfunktor für uniforme Limesräume, Math.Nachr. 46, 1-12 (197o)