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ON A PROBLEM OF KATĚTOV

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In his 1958 paper [1], M. Katětov studied extensions of locally finite covers. Consider the following statements.

(1) X is collectionwise normal and countably paracompact.

(2) Every locally finite open family in a closed subspace of X can be extended to a locally finite open family in X.

(3) Every locally finite functionally open family in a closed subspace of X can be extended to a locally finite open family in X.

(4) X is collectionwise normal.

In [K], Katětov proved $(1) \to (2) \to (4)$ and asked if the converses of these implications hold. Przymusiński recently answered one of Katětov's questions by showing $(2) \nrightarrow (1)$ under $V = L$. He also noticed $(2) \to (3) \to (4)$ and conjectured that $(4) \nrightarrow (3) \nrightarrow (2)$. In this paper we settle the remaining question of Katětov by showing that $(4) \nrightarrow (3) \nrightarrow (2)$. No set theoretic assumptions beyond the axiom of choice are needed, but we do show how extra assumptions can be used to strengthen the results. Details of results not proved here will appear in [3], along with related results of Przymusiński and the author. Included in [3] will be Przymusiński's work relating Katětov's properties to partitions of unity.

Definitions: If $\{V_\alpha | \alpha \in \varkappa\}$ is a family of subsets of K, a family $\{U_\alpha | \alpha \in \varkappa\}$ is said to extend $\{V_\alpha | \alpha \in \varkappa\}$ if $U_\alpha \cap K = V_\alpha$ for all $\alpha \in \varkappa$. A set is called functionally open (= cozero) in X if it can be represented as $\{x \in X | f(x) \neq 0\}$ for some continuous function $f: X \to R$. In a normal space, the functionally open sets are just the open F_σ sets. Przymusiński has defined a normal space X to be Katětov if it satisfies (2) above, and to be functionally Katětov if it satisfies (3) above. We will use Przymusiński's terminology throughout the rest of this paper.

§1: Functionally Katětov, not Katětov Spaces.

In this section we give two examples of spaces that are functionally Katětov but not Katětov.

Theorem 1: The Dowker space constructed by M. E. Rudin in [4] is functionally Katětov but not Katětov.

Theorem 2: $V = L$ implies there is a space that is functionally Katětov, locally countable, and of cardinality ω_1 but is not functionally Katětov.

We give Theorem 2 for two reasons. First, the construction for Theorem 2 is easier to work with than Rudin's construction (at least for those readers acquainted with the important technique used by Ostazewski in [2]). Second, $V = L$ allows us to build many nice properties into the example. Not only does it have the properties listed above, but (as in [2]) it can easily be modified to be locally compact, hereditarily separable, and first countable. We will prove Theorem 2 in detail since the proof will not be given in [3].

A Sketch of the proof of Theorem 1. We refer the reader to [4] for the construction of Rudin's example, X. To show that X is not Katětov, let $K = \{f \in X | f(n) = \omega_n \text{ for some } n\}$ and $V_n = \{f \in X | f(i) = \omega_i \text{ iff } i = n\}$ for each $n \in \omega$. Then K is closed in X and $\{V_n | n \in \omega\}$ is a locally finite open family in K that can be shown to have no locally finite open extension to X. The key to proving that X is functionally Katětov is the fact that every F_σ in X is closed. □

Proof of Theorem 2: Unless otherwise stated, α and β will denote countable ordinals, λ, ξ, γ, and η will denote limit ordinals less than or equal to ω_1, and n, m and k will denote non-negative integers. Thus a phrase such as "for all $\alpha, \lambda \in A$" should be read "for all $\alpha, \lambda \in A$ such that $\alpha < \omega_1$, $\lambda \leq \omega_1$, and λ is a limit ordinal".

Let $\{S_\lambda | \lambda < \omega_1\}$ satisfy

(1) for all λ, S_λ is a cofinal subset of λ that is cofinal in no smaller limit ordinal, and

(2) if S is an uncountable subset of ω_1, then there is a λ with $S_\lambda \subset S$.

Recall that $V = L$ implies that such a sequence exists.

The construction. Let $Y_\lambda = \omega \times \lambda$ for all λ. Let τ_ω be the discrete topology on Y_ω. For each λ, inductively construct τ_λ from $\{\tau_\xi : \xi < \lambda\}$ in the following way.

If λ is a limit of limit ordinals, let τ_λ be the topology generated by $\cup\{\tau_\xi | \xi < \lambda\}$.

If $\lambda = \xi + \omega$ for some ξ, partition S_ξ into $\omega \times \omega$ infinite pieces, $S_{n,m}$. Let τ_λ be the topology generated by all sets, U, of any of the following forms:

a) $U \in \tau_\xi$

b) $U = \{(0, \xi + m)\}$ for any m.

c) $U = (n, \xi + m) \cup V$ for any m and $V \in \tau_\xi$ such that either

(i) n is odd and V contains all but finitely many points of $\{0\} \times S_{n,m}$, or

(ii) n is positive and even and V contains all but finitely many points of $\{n-2, n-1, n\} \times S_{n,m}$.

Let $Y = Y_{\omega_1}$ and $\tau = \tau_{\omega_1}$. Then (Y, τ) is a functionally Katětov space that is not Katětov.

Verification of the properties of Y. For notational convenience, let $F_n = \{n\} \times \omega_1$ and set $\pi(A) = \{\alpha \mid (n,\alpha) \in A$ for some $n\}$ for each $A \subset Y$. Proofs of the first three facts below are left to the reader.

1) $n \times \alpha \in \tau_\lambda$ for all n,α and λ with $\alpha < \lambda$.

2) $F_0 \cup F_n \in \tau$ for each odd n.

3) $(n,\alpha) \in cl_{\tau_\xi}(\{m\} \times S_\eta)$ whenever $\eta \le \alpha < \xi$ and either

a) $0 < n$ is even and $m = n, n-1$ or $n-2$, or

b) n is odd and $m = 0$.

4) If for some ξ, V is clopen in τ_ξ and $\sup \pi V < \xi$, then V is clopen in τ_λ for each $\lambda > \xi$.

Proof: Let V and ξ be as in the hypothesis. For all $\lambda > \xi$, $\tau_\xi \subset \tau_\lambda$ and hence V is open in τ_λ. We prove by induction that V is also closed with respect to τ_λ. Fix $\lambda > \xi$ and suppose that V is clopen in τ_η for each η such that $\xi \le \eta < \lambda$.

If λ is a limit of limit ordinals, then $\bigcup_{\eta<\lambda}\tau_\eta$ is a base for τ_λ. Since $X_\eta - V \in \tau_\eta$ for all $\eta < \lambda$, $(X_\lambda - V) = \bigcup_{\eta<\lambda}(X_\eta - V) \in \tau_\lambda$ also.

If $\lambda = \eta + \omega$ for some η, notice that since $\sup V < \eta$, πV contains only finitely many points of S_η. Since V is closed in τ_η, $(Y_\eta - V) \cup \{(n,\eta+m)\} \in \tau_\lambda$ for all n and m. It follows that $Y_\lambda - V \in \tau_\lambda$. □

5) Y is regular.

Proof: We prove each τ_λ is regular by induction on λ. Note that τ_ω is regular. Fix λ and assume that for each $\xi < \lambda$, τ_ξ is a regular topology. Let $U \in \tau_\lambda$ and $x \in U$. We will show there is a clopen V with $x \in V \subset U$, and hence τ_λ is regular. There are two cases.

If λ is a limit of limit ordinals, then $x \in Y_\xi$ for some $\xi < \lambda$. Since τ_ξ is regular and Y_ξ is countable, there exists a clopen $V \in \tau_\xi$ with $x \in V \subset U$ and $\sup \pi V < \xi$. Then by (4) V is clopen in τ_λ also.

Now assume $\lambda = \xi + \omega$ for some ξ. Since τ_ξ satisfies (1), the set $\omega \times S_\xi$ is closed and discrete in X_ξ. Since X_ξ is countable and regular, there exists a closed discrete collection $\{V_{n,s} \mid n \in \omega, s \in S_\xi\}$ such that $(n,s) \in V_{n,s}$, each $V_{n,s}$ is clopen, and $V_{n,s} \cap V_{m,t} = \phi$ unless $n = m$ and $s = t$. In the light of the collection $\{V_{n,s} \mid n \in \omega, s \in S_\xi\}$, the regularity of τ_λ easily follows from the regularity of τ_ξ and the definition of τ_λ. □

6) Any two closed uncountable subsets of Y intersect.

Proof: Let H and K be uncountable closed subsets of Y. Then for some n, H has an uncountable intersection with $\{n\} \times \omega_1$ and hence contains $\{n\} \times S_\gamma$ for some γ. By (3-a), this implies that H contains all but countably many points of F_k, where k is the first even integer greater than n. By applying (3-a) repeatedly, it follows that H contains all but countably many points of F_j

for each even $j > k$. Since a similar statement holds for K, we have $H \cap K \neq \phi$.

7) Y is normal.

Proof: Let H and K be two disjoint closed subsets of Y. By (6), at least one of H and K is countable. Without loss of generality we assume H is countable. Choose $\lambda < \omega_1$ such that $\lambda > \sup \pi H$. Then since Y_λ is countable and regular, there is a V, clopen in τ_λ, such that $H \subset V$ and $V \cap K = \phi$. Moreover, by (1), V can be chosen so that $\sup \pi V < \lambda$. Then (4) implies that V is a clopen separation of H and K in Y. □

8) Y is functionally Katětov.

Proof: Assume K is closed in Y and $\mathcal{U}$ is a locally finite functionally open family of K. We must show that $\mathcal{U}$ has a locally finite open extension to Y. First note that $\mathcal{U}$ must be countable, for is not, choose $x_U \in U$ for each $U \in \mathcal{U}$. Since $\mathcal{U}$ is locally finite, $\{x_U | U \in \mathcal{U}\}$ is an uncountable closed discrete collection in Y. But every uncountable subset of Y contains $\{n\} \times S_\lambda$ for some n and λ, and hence is not closed discrete. Write $\mathcal{U}$ as $\{U_n | n \in \omega\}$.

Next observe that if any U_n is uncountable, then since it is functionally open, it contains an uncountable closed set, and hence $K - U_n$ is countable (by arguments similar to those used in (3-a) and (6)). Thus only finitely many of the U_n are uncountable and we can assume without loss of generality that none of the U_n are uncountable.

Choose $\xi > \sup(U_n)$ for all n with ξ so large that each $\{n\} \times (\omega_1 - \xi)$ has a neighborhood, W_n, in Y whose closure intersects only finitely many U_n. (The reader can check that such an ξ actually exists.) Y_ξ is regular and countable, and hence Katětov, so there exists a locally finite (in Y_ξ) open collection $\{V_n | n \in \omega\}$ such that $V_n \cap K = U_n$.
Let $V_n' = V_n - \cup\{\overline{W}_m | m < n$ and $\overline{W}_m \cap U_n = \phi\}$.
Then $\{V_n' | n \in \omega\}$ is the desired extension of $\{U_n | n \in \omega\}$. □

9) Y is not Katětov.

Proof: Let $K = Y - F_0$ and $\mathcal{U} = \{F_n | n$ is odd$\}$. By (1) and (2), $\mathcal{U}$ is a locally finite open family in the closed subspace K. If U is open in Y and $U \cap K = F_n$ for some odd n, then (3-b) implies that $F_0 - U$ is countable. It follows that the family $\mathcal{U}$ has no locally finite open extension to Y, and hence Y is not Katětov. □

§2: Collectionwise Normal, not Functionally Katětov Spaces.

In this section we modify Rudin's example, X, to produce an example, Z, of a space that is collectionwise normal but not functionally Katětov. The modification can be done on any "nice" Dowker space to produce an example. Using $V = L$, one can construct a collectionwise normal, not functionally Katětov space that is hereditarily separable, locally countable, and of cardinality ω_1.

Let $W_n = \{f \in X | f(i) = \omega_i$ when $i \le n$ and $cf(f(i)) \le \omega_n$ for all $i > n\}$ for each $n \ge 1$. Set $W = \cup\{W_n | n \ge 1\}$ and give W the subspace topology from X. For $n,m \in \omega$ and $A \subset W$, let $A^{n,m}$ denote $A \times (n,m)$ and A^n denote $A \times \{n\}$. Define

$$Z = W \cup \cup\{W_1^{n,m} | n,m \in \omega, m > 1\} \cup \cup\{W_k^m | k > 1, m \in \omega\}.$$

We generate a base for Z from the open sets in W. For each U, open in W, and each $n,m \in \omega$ with $m > 1$ and sequence of integers $\{k_i | i \in \omega\}$, the following two sets are declared to be basic open subsets of Z:

$$(U \cap W_1)^{n,m} \cup (U \cap W_m)^n$$

$$U \cup \cup\{(U \cap W_1)^{i,j} | i > k_j\} \cup \cup\{(U \cap W_j)^i | i > k_j\}.$$

Then Z is a collectionwise normal space that is not functionally Katětov. We will sketch the proof that Z has the desired properties; details will appear in [3].

To see that Z is not functionally Katětov, let $K = (W - W_1) \cup \cup\{W_m^n | n,m \in \omega, m > 1\}$ and set $V_m = \cup\{W_m^n | n \in \omega\}$ for each $m > 1$. Then $\{V_m | m > 1\}$ is a locally finite functionally open family in the closed subspace K that can be shown to have no locally finite open extension to Z.

The proof that Z is collectionwise normal is the hard part of this example. The following difficult fact is used repeatedly in the proof.

<u>Lemma</u>: For each n, $W_1 \cup W_n$ is collectionwise normal and countably paracompact.

§3: <u>Katětov Spaces that are not Countably Paracompact</u>.

We have already mentioned that Przymusiński has used $V = L$ to construct a Katětov space that is not countably paracompact. He does this by constructing a hereditarily normal, hereditarily separable Dowker space, and then showing that every such space is Katětov. It would be nice to have an example of a Katětov space that is not countably paracompact whose construction does not use set theoretic assumptions beyond the axiom of choice. The author conjectures that the space W (constructed in §2) is such a space. It is hoped that this conjecture will be settled in [3].

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