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A CLASS α AND COMPACTA WHICH ARE QUASI-HOMEOMORPHIC WITH SURFACES

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We shall consider metrizable spaces only. A map f of a compactum X into a space Y is said to be an ε-mapping if $\mathrm{diam}(f^{-1}(y))<\varepsilon$ for every $y\in f(X)$. Given two compact spaces X and Y, X is said to be Y-like if for every $\varepsilon>0$ there is an ε-mapping of X onto Y. The spaces X and Y are said to be quasi-homeomorphic if X is Y-like and Y is X-like. A compact space X is said to be quasi-embeddable into a space Y if for every $\varepsilon>0$ there is an ε-mapping of X into Y.

Mardešić and Segal [5] proved the following theorem: If X is a connected polyhedron then the following statements are equivalent:

(i) X is embeddable into S^2.

(ii) X is quasi-embeddable into S^2.

(iii) X does not contain any homeomorphic images of Kuratowski graphs K_1 and K_2 and any 2-umbrella.

Recall that K_1 is the 1-skelton of a 3-simplex with the mid--points of a pair of non-adjacent edges joined by a segment, K_2 is the 1-skelton of a 4-simplex. The n-umbrella is the one-point union of an n-ball and of an arc, relative to an interior point of the ball and an end-point of the arc.

I was looking for a bigger class such that the equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) hold for each member of this class and I have proved in [7] that this is the case for the class α defined as follows:

<u>Definition.</u> A locally connected continuum X belongs to the class α iff there is an $\varepsilon>0$ such that no simple closed curve $S\subset X$ with $\mathrm{diam}(S)<\varepsilon$ is a retract of X.

It is easy to see that the class α contains all compact, connected LC^1-spaces, and therefore it contains all compact, connected ANR-s too. Consequently, the equivalences (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) hold for each member X of these classes.

The following characterization of the connected planeable ANR-sets and of the planeable AR-sets follows: X is a connected planeable ANR iff $X \in \alpha$, X satisfies the condition (iii) and X is not homeomorphic with S^2. X is a planeable AR iff X is a locally connected continuum such that no simple closed curve $S \subset X$ is a retract of X, X satisfies the condition (iii) and X is not homeomorphic with S^2.

These results and the methods of the investigation of the class α as developed in [7] have been used to solve the following question raised by Mardešić and Segal in [6]: Is it true that any locally connected 2-dimensional compactum X which is M-like, where M is a surface (i.e. a closed 2-manifold), is homeomorphic with M?

I have proved that this is indeed the case. The proof will be published in detail in [8]. Here are the main ideas of the proof.

Ganea in [3] has proved this theorem under the additional assumption that X is an ANR. So we are going to prove only that our assumptions imply that X is an ANR. To do this - using the results concerning the class α, as mentioned on the beginning - we prove the following main lemmas:

Lemma 1. Let Y be a compact space such that the group $H_1(Y)$ (the Čech homology group with integer coefficients) is finitely generated. If X is a locally connected continuum which is Y-like, then $X \in \alpha$.

Lemma 2. Assume that $X \in \alpha$ and that for every $\lambda > 0$ there is a set $A \subset X$ homeomorphic either with K_1 or with K_2 and such that $\mathrm{diam}(A) < \lambda$. Then for each k=1,2,... there is a sequence $B_1, \ldots, B_k$ of disjoint subsets of X, each of which is homeomorphic either with K_1 or with K_2.

Lemma 3. Assume that $X \in \alpha$, X does not contain any 2-umbrella and there is a $\lambda > 0$ such that there is no set $A \subset X$ with $\mathrm{diam}(A) < \lambda$ homeomorphic either with K_1 or with K_2. Then for every $x \in X$ there is a neighborhood of x in X being a compact planeable AR-set.

Using the Bennet's result [1] that the 2-umbrella is not quasi-embeddable in S^2 and E^2 and using the theory of the universal covering spaces we proved also the following lemma:

Lemma 4. The 2-umbrella is not quasi-embeddable in any 2-manifold.

Now, we proceed as follows: Assume that X is a locally connected compactum, dim $X \geq 2$ and X is M-like, where M is a surface. Since M is connected and for every $\varepsilon > 0$ there is an ε-mapping of X onto M, it follows that X is connected. Now, lemma 1 implies that $X \in \alpha$. Since dim M=2 and ε-mappings cannot diminish the dimension, it follows that dim X =2. By lemma 4, X does not contain any 2-umbrella. Assume that for every $\lambda > 0$ there is a set $A \subset X$ with diam$(A) < \lambda$ homeomorphic either with K_1 or with K_2. It has been proved by Borsuk in [2] that the surface M does not contain any subset which is the union of $k = \gamma(M) + 1$ components, each of which is homeomorphic either with K_1 or with K_2, where $\gamma(M)$ denotes the genus of M. Using lemma 2 and the fact that each ε-mapping with sufficiently small $\varepsilon > 0$ maps the given sequence of disjoint compact sets onto a sequence of disjoint sets, we obtain a contradiction with the Borsuk's result. Consequently, X satisfies all the assumptions of lemma 3, and therefore (by Hanner theorem) X is an ANR. Using now the Ganea's result [3] mentioned above, we conclude:

Theorem 1. If X is a locally connected compactum, dim $X \geq 2$ and X is M-like, where M is a surface, then X is homeomorphic with M.

This implies easily the following

Corollary. If X is a compactum quasi-homeomorphic with a surface M then X is homeomorphic with M.

Using similar methods we proved the following theorem, which generalizes the Borsuk's result [2] that each locally planeable ANR-set is embeddable into a surface.

Theorem 2. Each locally planeable space $X \in \alpha$ is embeddable into a surface.

Lemma 4 leads naturally to the following question:

Question 1. Is it true that the n-umbrella is not quasi-embeddable in any n-manifold?

Note, that it has been proved by Mardešić and Segal in [5] that the n-umbrella is not quasi-embeddable into S^n and E^n. However, our method of the proof of lemma 4 gives the positive answer to question 1 only if we know that the universal covering space for a given n-manifold is either E^n or S^n (or is embeddable into S^n).

The following other questions concerning quasi-homeomorphisms appear under the investigation of this subject:

Question 2. Is any crumpled n-cube quasi-homeomorphic with the usual n-cube I^n? Or, if not, is it I^n-like?

Recall that the crumpled n-cube is the closure of a component of $S^n \setminus S$, where S is any (n-1)-sphere topologically embeded in S^n. The question concerning crumpled cubes is closely related to the next one, which is suggested by the Ganea's example [4] of a 3-dimensional ANR-set, which is quasi-homeomorphic with S^3, but not homeomorphic with S^3.

Question 3. Is any decomposition space of S^n such that the non-degenerate elements are simple arcs (or AR-sets, or even compact sets with the trivial shape) quasi-homeomorphic with S^n? Or, if not, is it S^n-like? The same question concerns the decomposition space of any n-manifold.

The following question is known to many people, but seems to be a difficult one:

Question 4. Let N and M be two compact n-manifolds such that N is M-like. Is it true that N is homeomorphic with M?

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