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ON QUASI-UNIFORM CONVERGENCE

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In this paper we extend the classical theorem that uniform convergence of a sequence of continuous functions implies the continuity of the limit function. This is accomplished by means of sequentially complete, $\mathcal{U}$-Cauchy sequences and quasi-uniform convergence.

Let X be a nonempty set. A *quasi-uniformity* for X is a filter $\mathcal{U}$ of reflexive subsets of $X \times X$ such that if $U \in \mathcal{U}$, there is $V \in \mathcal{U}$ such that $V \circ V \subset U$ [4]. A quasi-uniform space $(X,\mathcal{U})$ is *complete* if every $\mathcal{U}$-Cauchy filter converges [4]. Let (X,τ) be a topological space. For each $A \in \tau$, let $S_A = (A \times A) \cup (X - A) \times X$ and $S = \{S_A : A \in \tau\}$. Then S is a subbase for a compatible quasi-uniformity on X called *Pervin quasi-uniformity* [4]. A quasi-uniform space $(X,\mathcal{U})$ is R_3, if, given $x \in X$ and $U \in \mathcal{U}$, there exists a symmetric $W \in \mathcal{U}$ such that $W \circ W(x) \subset U(x)$ [3]. It is known that a topological space admits a compatible R_3 quasi-uniformity if and only if it is regular [4, Theorem 3.17]. We get the following theorem as a consequence of the above definition.

THEOREM 1. *Let* $(X,\mathcal{U})$ *be an* R_3 *quasi-uniform space. Let* $x \in X$ *and* $U \in \mathcal{U}$. *Then for each positive integer* n *there exists a symmetric entourage* $V \in \mathcal{U}$ *such that* $V^n(x) \subset U(x)$.

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DEFINITION. Let $(X,\mathcal{U})$ be a quasi-uniform space. A sequence $(x_i)_{i=1}^{\infty}$ in X is said to be $\mathcal{U}$-Cauchy if for each $V \in \mathcal{U}$ there exists a positive integer n such that for all $i > n$, $x_i \in V(x_n)$.

DEFINITION. A quasi-uniform space is said to be sequentially complete if every $\mathcal{U}$-Cauchy sequence converges to a point in X.

THEOREM 2. *Let $(X,\mathcal{U})$ be a complete quasi-uniform space. Then $(X,\mathcal{U})$ is sequentially complete.*

PROOF: Let $(x_i)_{i=1}^{\infty}$ be a $\mathcal{U}$-Cauchy sequence in X. For each positive integer n let $F_n = \{x_i\}_{i=n}^{\infty}$. Let f be a filter generated by $\{F_n$: n is a positive integer$\}$. Clearly f is a $\mathcal{U}$-Cauchy filter. By hypothesis there exists a $y \in X$ such that f converges to y. Consequently for each $U \in \mathcal{U}$, there exists an $F_n \in f$ such that $F_n \subset U(y)$. Therefore for all positive integers $i > n$, $x_i \in U(y)$. Thus $(x_i)_{i=1}^{\infty}$ converges to y.

DEFINITION [2]. A space X is sequentially compact if and only if every sequence in X has a subsequence that converges to a point in X.

DEFINITION [1]. A (sub)base β for a quasi-uniformity $\mathcal{U}$ is transitive provided that for each $B \in \beta$, $B \circ B = B$. A quasi-uniformity with a transitive base is called a transitive quasi-uniformity.

THEOREM 3. *Every R_3 transitive quasi-uniformity of a sequentially compact space is sequentially complete.*

PROOF. Let (X,τ) be a sequentially compact space and let $\mathcal{U}$ be a compatible locally symmetric and transitive quasi-uniformity on X. Let $(x_i)_{i=1}^{\infty}$ be a $\mathcal{U}$-Cauchy sequence and $(x_{ij})_{j=1}^{\infty}$ be a subsequence that converges to some point p. Let G be an open set containing p. By hypothesis there exist W and V $\in \mathcal{U}$ such that $V = V \circ V$, and $V(p) \subset G$ and $W = W^{-1}$ and $W \circ W(p) \subset V(p) \subset G$. Since $(x_i)_{i=1}^{\infty}$ is $\mathcal{U}$-Cauchy there exists a positive integer m such that for all $i > m$, $x_i \in W(x_n)$ and for all $j \geq m$, $x_{ij} \in W(p)$. Then $p \in W(x_{ij}) \subset W \circ W(x_n)$. Thus $x_n \in W \circ W(p)$ so that $V(x_n) \subset V \circ V(p) \subset G$, and for $i > m$, $x_i \in V(x_n)$ $\subset G$. Consequently $\{x_i\}_{i=1}^{\infty}$ converges to p.

It is natural to investigate whether the converse of the Theorem 2 holds. Next we give an example of a countably compact, first countable Hausdorff space which is sequentially complete but not complete.

We know that the Pervin quasi-uniformity is precompact and transitive and that a quasi-uniform space is compact if and only if it is complete and precompact [4, Theorem 4.14]. Let $(\mathcal{O},w)$ be the space of all ordinals less than the first uncountable ordinal. It is known that $(\mathcal{O},w)$ is a sequentially compact, first countable space that is not Lindelöf (and hence not compact) [2, Example 8.16]. Let P be the Pervin quasi-uniformity on $(\mathcal{O},w)$. By Theorem 3, P is sequentially complete. However, P cannot be complete since P is precompact and P is not compact.

THEOREM 4. <u>Every closed subspace of sequentially complete space is sequentially complete</u>.

THEOREM 5. <u>Let (X_α, U_α) be any collection of sequentially complete quasi-uniform spaces, then the product quasi-uniformity of the product space is sequentially complete</u>.

DEFINITION. A sequence of functions $(f_i)_{i=1}^{\infty}$ from a topological space X into a quasi-uniform space $(Y,\mathcal{U})$ is a $\mathcal{U}$-<u>Cauchy sequence</u> if for each $U \in \mathcal{U}$ there exists a positive integer n (depending on U) such that for each $x \in X$ and for each $m > n$, $(f_n(x), f_m(x)) \in U$.

DEFINITION. Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions from a topological space (X,τ) into a quasi-uniform space $(Y,\mathcal{U})$. Then $(f_n)_{n=1}^{\infty}$ is said to converge quasi-uniformly if there exists a function $g: X \to Y$, such that for each $U \in \mathcal{U}$ there exists a positive integer N (depending on U) such that for each n, $n > N$ and each $x \in X$, $(g(x), f_n(x)) \in U$.

THEOREM 6. *Let $(f_i)_{i=1}^{\infty}$ be a sequence of functions from a topological space X into a Hausdorff sequentially complete quasi-uniform space $(Y,\mathcal{U})$ such that $(f_i)_{i=1}^{\infty}$ is $\mathcal{U}$-Cauchy. Then $(f_i)_{i=1}^{\infty}$ converges quasi-uniformly.*

PROOF: By hypothesis for each $x \in X$, $(f_i(x))_{i=1}^{\infty}$ is a $\mathcal{U}$-Cauchy sequence. Let $x \in X$ and $y \in Y$ such that limit $(f_i(x))_{i=1}^{\infty} = y$. Define $f: X \to Y$ by $f(x) = y = \text{limit } (f_i(x))_{i=1}^{\infty}$. Clearly $(f_i)_{i=1}^{\infty}$ converges quasi-uniformly to f.

THEOREM 7. *Let $(f_n)_{n=1}^{\infty}$ be a sequence of continuous functions from a topological space (X,τ) into an R_3 quasi-uniform space $(y,\mathcal{U})$ such that $(f_n)_{n=1}^{\infty}$ converges quasi-uniformly to a function $g: X \to Y$. Then g is continuous.*

PROOF. Let $x \in X$ and $U \in \mathcal{U}$. Let W be a symmetric entourage such that $W \circ W \circ W(g(x)) \subset U(g(x))$. Let N be a positive integer such that for all $n > N$ and for each $z \in X$, $(g(z), f_n(z)) \in W$. Let $n > N$ and let $y \in f_n^{-1}(W(f_n(x)))$; then $(f_n(x), f_n(y)) \in W$. We also have that

$(g(y), f_n(y))$, $(g(x), f_n(x)) \in W$ so that $(g(x), g(y)) \in W \circ W \circ W$. Thus $g(y) \in W \circ W \circ W(g(x)) \subset U(g(x))$. Therefore g is a continuous function.

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