LINEAR-TOPOLOGICAL PROPERTIES OF OPERATOR SPACES

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Absolutely summing and integral operators play an important role in the theory of Banach spaces. In this paper the Banach spaces $\Pi_p(E,F)$ and $I_p(E,F)$ of absolutely p-summing and p-integral operators are studied from the following point of view: What can be said about the linear-topological structure of $\Pi_p(E,F)$ and $I_p(E,F)$ if the structure of E (or E' -the topological dual) and F is known. In recent years several authors dealt with questions of this type (see e.g. [2], [3], [7], and [9]).

First we give the basic definitions (see [8]). Let E and F be Banach spaces and p a real number, $1 \le p < \infty$. A linear operator $T : E \longrightarrow F$ is called absolutely p-summing whenever there exists a constant K such that for all finite subsets $\{x_k\} \subset E$ the following inequality holds

$$\left(\sum_k \|Tx_k\|^p \right)^{1/p} \le K \sup_{\substack{f\in E' \\ \|f\|=1}} \left(\sum_k |f(x_k)|^p \right)^{1/p}$$

The absolutely p-summing norm $\pi_p(T)$ is the smallest constant K satisfying the above inequality. A linear operator $T : E \longrightarrow F$ is called p-integral if there is a positive measure μ defined on the weak-star compact unit ball U^o of E' such that jT admits the following factorization:

$$jT : E \xrightarrow{I} C(U^o) \xrightarrow{J} L_p(U^o, \mu) \xrightarrow{S} F'' ,$$

where I, J, and $j : F \longrightarrow F''$ are the corresponding canonical embeddings and S is a linear operator with $\|S\| \le 1$. The p-integral norm is defined by

$$i_p(T) = \inf \mu(U^o)^{1/p}$$

where the infimum is taken over all possible factorizations. The spaces of absolutely p-summing and p-integral operators are Banach spaces, denoted by $\Pi_p(E,F)$ and $I_p(E,F)$, respectively. One of the first results on the linear-topological structure of Π_p and I_p is the following theorem (see e.g. [4] for the definitions of approximation properties).

<u>Theorem 1</u> ([3] , [9]). Suppose E has the bounded approximation property and let $1 \leq p < \infty$. If E and F are reflexive, then

(a) $\Pi_p(E,F)$ is reflexive, and

(b) $I_p(E,F)$ is reflexive provided $1 < p < \infty$.

In the sequel we consider properties weaker than reflexivity. Recall that a Banach space E posesses the Radon-Nikodým property if every countably additive E-valued measure of finite variation has a Bochner derivative with respect to its variation.

<u>Theorem 2</u> ([5]). Let E and F be Banach spaces such that E' and F have the bounded approximation property and let $1 \leq p < \infty$. If E' and F posess the Radon-Nikodým property, then

(a) $\Pi_p(E,F)$ has the Radon-Nikodým property, and

(b) $I_p(E,F)$ has the Radon-Nikodým property provided E is WCG.

Banach spaces that do not contain a subspace isomorphic to c_o (the space of scalar sequences converging to zero) posess important properties (see e.g. [1]). We shall use the notation $E \not\supset c_o$. Furthermore denote the spaces of p-nuclear and quasi-p-nuclear operators by $N_p(E,F)$ and $N_p^Q(E,F)$, respectively (see [8] for the definition).

<u>Theorem 3</u> ([5]). Let E and F be Banach spaces such that E' and F posess the bounded approximation property and let $1 \leq p < \infty$. Suppose $E' \not\supset c_o$ and $F \not\supset c_o$.

(a) If $\Pi_p(E,F) = N_p^Q(E,F)$, then $\Pi_p(E,F) \not\supset c_o$.

(b) If $I_p(E,F) = N_p(E,F)$, then $I_p(E,F) \not\supset c_o$.

Examples show that the assumptions $\Pi_p = N_p^Q$ and $I_p = N_p$ are essential at least for $1 < p < \infty$.

Now recall that a Banach space is said to be weakly sequentially complete if every weak Cauchy sequence converges.

<u>Theorem 4</u> ([6]). Let E be a Banach space with an unconditional basis and let $1 \leq p < \infty$. If E' and F are weakly sequentially complete, then $\Pi_p(E,F)$ and $I_p(E,F)$ are weakly sequentially complete, too.

As in the preceding case, examples show that the existence of an unconditional basis is essential at least for $1 < p < \infty$.

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