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In: Josef Novák (ed.): General topology and its relations to modern analysis and algebra IV, Proceedings of the fourth Prague topological symposium, 1976, Part B: Contributed Papers. Society of Czechoslovak Mathematicians and Physicist, Praha, 1977. pp. [288]--290.

Persistent URL: http://dml.cz/dmlcz/700622

COMPACT HAUSDORFF SPACES WITH TWO OPEN SETS

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Alan H. Schoenfeld and Gary Gruenhage [7] have shown that if a compact metric infinite space X has, up to a homeomorphism, only two open non-empty subsets, then it is homeomorphic to the Cantor set, by proving that (1) X does not have isolated points, and that (2) X is totally disconnected. For the brevity, the phrase the space with two open sets will be used. We discuss here (infinite) spaces with two open sets which are compact and Hausdorff, however not necessarily metric.

We show that if X is a compact Hausdorff (infinite) space with two open sets, then (3) X has a countable base around each closed (non-open) subset, (4) X has the Souslin property hereditarily, still having properties (1) and (2). The space called "the double arrow", described in Mémoire by Alexandroff and Urysohn, 1929, is a compact Hausdorff separable non-metric space with two open sets.

When the space S is not separable, then it is nowhere separable, and we show that (5) X is the union of an increasing sequence of $\aleph_1$ nowhere dense separable closed subsets; from this immediately follows that (6) X contains a dense subset of cardinality $\aleph_1$, and that (7) X has cardinality that of continuum. Such a space, having the Souslin property and the property (5), cannot be constructed within ZFC, since the existence of such a space contradicts the axioms of the theory ZFC + ($\aleph_1 < 2^{\aleph_0}$) + Martin's axiom. However, if there exist homogeneous Souslin lines, then compact Hausdorff non-separable spaces with two open sets can be constructed in the same way as "the double arrow" is constructed from the real line; the existence of homogeneous Souslin lines can be shown in the theory ZFC + $\diamondsuit$ -hypothesis (see the book by Devlin and Johnsbråten [2] , p. 39). Thus, the existence of compact Hausdorff non-separable spaces with two open sets cannot be also disproved within ZFC. References to the consistency results needed here can be found in the book by Jech [3] and in the book [2] , loco cit.

Conjectures. 1. The strongest one is that the compact Hausdorff spaces with two open sets are ordered ones; this means, in particular, that they are Souslin spaces if they are not separable. 2. A weaker conjecture: the existence of compact Hausdorff non-separable spaces with two open sets implies the existence of Souslin spaces, i.e. contradicts the Souslin hypothesis. 3. The following conjecture seems to be weaker than the preceding one: the existence of compact Hausdorff non-separable spaces with two open sets contradicts the Devlin's hypothesis from [1] : a compact Hausdorff space in which any countable union of nowhere dense subsets is again nowhere dense cannot be the union of $\aleph_1$ nowhere dense subsets. To prove that conjecture it would be sufficient to check, in view of (5), the assumptions of Devlin's hypothesis for our spaces, for instance that nowhere dense subspaces are separable. Note that the Devlin's hypothesis implies the Souslin hypothesis, as the last one can be stated as follows (cf. Papić [4]): an ordered space with the Souslin property cannot be the union of an increasing sequence of $\aleph_1$ nowhere dense subsets.

We shall sketch the way to the results announced above. The full proofs will be published in Colloquium Mathematicum.

Let X be a compact Hausdorff (infinite) space with two open sets. One of these sets is compact, being homeomorphic to X. The second one is non-compact, being homeomorphic to each of the subspaces $X - \{x\}$; this follows from the fact that X does not have isolated points, X being infinite. As in the metric case, X is totally disconnected; the proof of this fact given in [7] remains valid, because the known Moore theorem on the existence in each non-degenerate metric continuum of at least two non-cutpoints, used in that proof, is valid for Hausdorff continua, too.

The following simple observation is the key for obtaining the remaining properties: if U is a non-compact open subset of X, then $U = U_1 \cup U_2 \cup \ldots$, where U_i are closed-open, mutually disjoint and non-empty. From this we get immediately that if F is a closed non-open subset of X, then there exists a countable base around F consisting of closed-open subsets (a more detailed version of (3)); in particular, X is first countable. We get as a corollary that X has the Souslin property hereditarily; for this way of getting the last property, directly from the preceding one, the author is indebted to a comment by Professor I. Juhász during the Symposium.

If X is not separable, then each non-empty closed-open subset of X is not separable, being homeomorphic to X. Since the closed-open subsets form a base, each separable subspace of X is nowhere dense. The main result, a more detailed version of (5), is the following

<u>Theorem</u>. If X is a compact Hausdorff non-separable space with two open sets, then $X = \bigcup \{F_\alpha : \alpha < \omega_1\}$, where F_α are closed separable subsets of X such that F_α is a nowhere dense subset of F_β, if $\alpha < \beta$.

The proof consists of an inductive construction of the sets F_α, using the properties of X proved before, mainly the property (3). The construction resembles known from a long time constructions on Souslin lines (see [2], p. 13), as well as more recent ones leading to estimations of cardinalities of spaces as in the papers by Ponomarev [6], Pol [5] and Šapirovskii [8].

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