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AN APPLICATION OF MODULAR SPACES TO INTEGRAL EQUATIONS

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1. Let (Ω, Σ, μ) be a measure space, μ finite, and let $\mathcal{X}$ be the space of real-valued, Σ-measurable functions on Ω with equality μ-a.e. Let a function $k: \Omega \times \Omega \times [0, \infty) \to [0, \infty)$ (called in the sequel the kernel) be measurable in $\Omega \times \Omega \times [0, \infty)$, $k(t,s,u)$ continuous and convex as a function of $u \in [0, \infty)$ for all $(t,s) \in \Omega \times \Omega$, $k(t,s,u)=0$ iff $u = 0$. The following integral equation may be considered :

$$(1) \qquad x(t) = \varkappa \int_{\Omega} k(t,s,|x(s)|)\, d\mu(s) + x_0(t).$$

Now, one investigates usually solutions of this equation belonging to a fixed function space, as $L^p(\Omega, \Sigma, \mu)$ or the space of continuous functions $C(\Omega)$ in case Ω is a compact topological space. The aim of the results presented here is to consider the solutions of the above equation as elements of a certain space X_{ϱ_s} which depends on the kernel k. The treatment may be generalized, namely, one may observe that the integral at the right-hand side is a modular, as considered in the theory of modular spaces. The general theory of modular spaces depending on a parameter, as needed here, was presented at the Third Prague Topological Symposium 1971 [4] by A.Waszak and myself. We shall adopt here the notation introduced in [4]. The investigation of modular equations is due to T.M. Jędryka and myself ([1], [2]).

2. Let $\varrho : \Omega \times \mathcal{X} \to [0, \infty]$ be a family of convex modulars on $\mathcal{X}$, i.e. $\varrho(t,x) \geqslant 0$, $\varrho(t,x) = 0$ μ-a.e. implies $x = 0$, $\varrho(t,-x) = \varrho(t,x)$, $\varrho(t, \alpha x + \beta y) \leqslant \alpha \varrho(t,x) + \beta \varrho(t,y)$ for $\alpha, \beta \geqslant 0$, $\alpha + \beta = 1$, and $\varrho(t,x)$ is Σ-measurable in the variable $t \in \Omega$ for all $x \in \mathcal{X}$. We denote by X the set of all $x \in \mathcal{X}$ such that $\varrho(t, \lambda x) \to 0$ as $\lambda \to 0$ μ-a.e. in Ω and we restrict ϱ to the product $\Omega \times X$. Then $\varrho_s(x) = \int_{\Omega} \varrho(t,x)\, d\mu$ is a modular in X and

$$X_{\varrho_s} = \{x : x \in X, \ \varrho_s(\lambda x) \to 0 \ \text{as} \ \lambda \to 0\}$$

is the modular space generated by means of the modular ϱ_s. It follows from the definition of X_{ϱ_s} that an element $x \in X$ belongs to X_{ϱ_s} iff there exists a number $\lambda_0 > 0$ such that $\varrho_s(\lambda_0 x) < \infty$. The space X_{ϱ_s} is a normed space with norm

$$\|x\|_{\varrho_s} = \inf \{u > 0: \varrho_s(x/u) \le 1\}.$$

Now, let $I: \Omega \times \mathfrak{X} \to [-\infty, \infty]$ be a functional such that $\varrho(t,x) = |I(t,x)|$ satisfies all the above assumptions. Our purpose is to investigate the equations

$$x(t) = \varkappa I(t,x) \quad \text{and} \quad x(t) = \varkappa I(t,x) + x_0(t) , \quad -\text{a.e.},$$

where $\varkappa \neq 0$ is a given number and x_0 is a given fixed element of X_{ϱ_s}. We consider operators A and B defined by

$$(A(x))(t) = \varkappa I(t,x) \quad \text{and} \quad (B(x))(t) = \varkappa I(t,x) + x_0(t).$$

Solutions of the above equations are fixed point of operators A and B, respectively. We are going to find sufficient conditions in order that A and B be contraction operators in X_{ϱ_s} or in the ball

$$K_{\varrho_s}(r) = \{x: \ x \in X_{\varrho_s} , \|x\|_{\varrho_s} \le r\}.$$

This will make possible, in case when X_{ϱ_s} is complete, to formulate theorems on existence and uniqueness of the solution of the above equations.

3. We give now propositions concerning operators A and B in the general case.

Proposition 3.1. (a) If for every $x \in X_{\varrho_s}$ and every $\lambda_1 > 0$ there exist numbers $C > 0$ and $\lambda_2 > 0$ such that

(2) $$\varrho(t, \lambda_2 \varrho(\cdot, x)) \le C \varrho(t, \lambda_1 x) \quad \mu\text{-a.e. in } \Omega ,$$

then both A and B map X_{ϱ_s} into itself.

(b) Let $0 < r < \infty$, $0 < R < \infty$. If for every $x \in X_{\varrho_s}$ and every λ such that $0 < \lambda \le 1/R$ there holds the inequality

(3) $$\varrho(t, \lambda \varkappa \varrho(\cdot, x)) \le \varrho\left(t, \lambda \frac{R}{r} x\right) \quad \mu\text{-a.e. in } \Omega ,$$

then A maps $K_{\varrho_s}(r)$ in $K_{\varrho_s}(R)$. If, moreover, $R = (1 - \vartheta)r$, where $0 < \vartheta < 1$, and $\|x_0\|_{\varrho_s} \le \vartheta r$, then B maps $K_{\varrho_s}(r)$ into itself.

Proof. (a) Integrating the inequality (2) over Ω we obtain

$$\varrho_s(\lambda_2 \varkappa^{-1} A(x)) = \varrho_s(\lambda_2 \varrho(\cdot, x)) \le C \varrho_s(\lambda_1 x).$$

Hence $x \in X_{\varrho_s}$ implies $A(x) \in X_{\varrho_s}$.

(b) Integrating the inequality (3) over Ω we get $\varrho_s(\lambda A(x)) \leq \leq \varrho_s(\lambda R r^{-1} x)$. Taking $\lambda = 1/R$ we obtain $\varrho_s(A(x)/R) \leq \varrho_s(x/r)$. Thus, $x \in K_{\varrho_s}(r)$ implies $A(x) \in K_{\varrho_s}(R)$. Now, if $R = (1-\vartheta)r$, then

$$\|B(x)\|_{\varrho_s} \leq \|A(x)\|_{\varrho_s} + \|x_0\|_{\varrho_s} \leq (1-\vartheta)r + \vartheta r = r \quad ,$$

i.e. B maps $K_{\varrho_s}(r)$ into itself.

<u>Proposition 3.2.</u>(a) Let ϱ satisfy the condition 3.1(b) with $R = r$. Moreover, let us suppose that for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $\eta > 0$ and all $x, y \in K_{\varrho_s}(r)$ there holds the inequality

$$(4) \qquad \int_\Omega \varrho\left(t, \frac{I(\cdot,x) - I(\cdot,y)}{\eta\varepsilon}\right) d\mu \leq \int_\Omega \varrho\left(t, \frac{x-y}{\varkappa\eta\delta}\right) d\mu.$$

Then A maps $K_{\varrho_s}(r)$ into itself, continuously. This remains true for $r = \infty$, where $K_{\varrho_s}(\infty) = X_{\varrho_s}$.

(b) Let $\|x_0\|_{\varrho_s} \leq \vartheta r$, $0 < \vartheta < 1$, and let ϱ satisfy the condition 3.1(b) with $R = (1-\vartheta)r$. Moreover, let us suppose that for every $\varepsilon > 0$ there exists a number $\delta > 0$ such that for every $\eta > 0$ and for all $x, y \in K_{\varrho_s}(r)$ there holds the inequality (4). Then B maps $K_{\varrho_s}(r)$ into itself, continuously. This remains true for $r = \infty$, where $K_{\varrho_s}(\infty) = X_{\varrho_s}$.

<u>Proposition 3.3.</u>(a) Let ϱ satisfy the condition 3.1(b) with $R = r$. Moreover, let us suppose that there exists a number $\alpha > 0$ such that for every $\eta > 0$ and all $x, y \in K_{\varrho_s}(r)$ there holds the inequality

$$(5) \qquad \int_\Omega \varrho\left(t, \frac{I(\cdot,x) - I(\cdot,y)}{\eta}\right) d\mu \leq \int_\Omega \varrho\left(t, \frac{\alpha(x-y)}{\varkappa\eta}\right) d\mu.$$

Then $\|A(x) - A(y)\|_{\varrho_s} \leq \alpha \|x-y\|_{\varrho_s}$ for all $x, y \in K_{\varrho_s}(r)$. This remains also true for $r = \infty$. If $0 < \alpha < 1$, A is a contraction operator in $K_{\varrho_s}(r)$.

(b) Let $\|x_0\|_{\varrho_s} \leq \vartheta r$, $0 < \vartheta < 1$, and let ϱ satisfy the condition 3.1(b) with $R = (1-\vartheta)r$. Moreover, let us suppose that there exists a number $\alpha > 0$ such that for every $\eta > 0$ and all $x, y \in K_{\varrho_s}(r)$ there holds the inequality (5). Then $\|B(x) - B(y)\|_{\varrho_s} \leq \alpha \|x-y\|_{\varrho_s}$ for all $x, y \in K_{\varrho_s}(r)$. This remains true also in case $r = \infty$. If $0 < \alpha < 1$, then B is a contraction operator in $K_{\varrho_s}(r)$.

We limit ourselves to the proof of 3.3(a). Indeed, we have

$$\|A(x) - A(y)\|_{\rho_s} = |\varkappa| \cdot \inf \left\{ \eta > 0 : \int_{\Omega} \rho\left(t, \frac{I(\cdot,x) - I(\cdot,y)}{\eta}\right) d\mu \leq 1 \right\} \leq$$

$$\leq |\varkappa| \cdot \inf \left\{ \eta > 0 : \int_{\Omega} \rho\left(t, \frac{\alpha(x - y)}{\varkappa \eta}\right) d\mu \leq 1 \right\} = \alpha \|x - y\|_{\rho_s} .$$

4. In order to apply the above considerations to the integral equation (1), we take

$$(6) \qquad I(t,x) = \int_{\Omega} k(t,s,|x(s)|) d\mu(s).$$

Under the assumptions on k formulated in 1., $\rho(t,x) = I(t,x)$ satisfies the assumptions from 2. Hence we may apply the Propositions from 3. However, in order to make use of the Banach fixed-point theorem, we must know that the respective modular space X_{ρ_s} is complete in the norm $\| \ \|_{\rho_s}$. The following theorem is true (see [1]):

<u>Theorem 4.1.</u> If for every $u > 0$ there holds the inequality

$$\int_{\Omega} k(t,s,u) d\mu(t) > 0$$

for μ-a.e. $s \in \Omega$, then the space X_{ρ_s} with norm $\| \ \|_{\rho_s}$ is a Banach space.

Proof. Special case of this theorem when $k(t,s,u)$ is independent of t was given in [3], 2.31. The present proof (see also [1]) runs similar lines. First, we observe that if a function $f: \Omega \to [0,\infty)$ is Σ-measurable and positive μ-a.e., then the measure μ is ν-absolutely continuous, where $\nu(A) = \int_A f(s) d\mu(s)$. Thus, taking $\varepsilon > 0$ and $f(s) = \int_{\Omega} k(t,s,\varepsilon) d\mu(t)$, there exists a number $\eta > 0$ such that $\nu(A) < \eta$, $A \in \Sigma$, imply $\mu(A) < \varepsilon$. Let (x_n) be a Cauchy sequence in X_{ρ_s} and let us take any $\lambda > 0$, then

$$\rho_s(\lambda(x_n - x_m)) \to 0 \quad \text{as} \quad m,n \to \infty .$$

There exists an N such that $\rho_s(\lambda(x_n - x_m)) < \eta$ for $m,n > N$. Denoting $B_{m,n} = \{s \in \Omega : \lambda|x_n(s) - x_m(s)| \geq \varepsilon\}$, we obtain

$$\nu(B_{m,n}) = \int_{B_{m,n}} \left\{ \int_{\Omega} k(t,s,\varepsilon) d\mu(t) \right\} d\mu(s) \leq \rho_s(\lambda(x_n - x_m)) < \eta ,$$

and so $\mu(B_{m,n}) < \varepsilon$ for $m,n > N$. Consequently, (λx_n) tends to a function $x_\lambda \in \mathfrak{X}$ in μ-measure in Ω. It is easily observed that x_λ is of the form $x_\lambda = \lambda x$. Standard application of Fatou lemma shows that $\rho_s(\lambda(x_n - x)) \to 0$ as $n \to \infty$, i.e. $\|x_n - x\|_{\rho_s} \to 0$ as $n \to \infty$.

5. Now, we shall adopt the assumptions of Propositions 3.1-3.3 to the case of the modular $\varrho(t,x)= I(t,x)$ defined by (6). Operators A and B are then defined as in 2. Let us write

$$k_1^\lambda(t,u,v)= \int_\Omega k[t,s,\ \lambda k(s,u,v)]\,d\mu(s),$$

$$\varrho_1^\lambda(t,x) = \int_\Omega k_1^\lambda(t,s,\ |x(s)|)\,d\mu(s).$$

Proposition 5.1. (a) Let $0<r<\infty$, $0<R<\infty$ and let us suppose that for every $x\in K_{\varrho_s}(r)$ and every λ such that $0<\lambda\leq 1/R$ there holds the inequality

$$\varrho_1^\lambda(t,x)\leq \mu(\Omega)\varrho\left(t,\ \lambda\frac{R}{r}x\right)\quad \mu\text{-a.e. in } \Omega .$$

Then A maps $K_{\varrho_s}(r)$ in $K_{\varrho_s}(R)$ for every $\varkappa$ such that $0<|\varkappa|<1/\mu(\Omega)$.

(b) Let $0<r<\infty$ and $\|x_0\|_{\varrho_s}\leq\vartheta r$, where $0<\vartheta<1$. If for every $x\in K_{\varrho_s}(r)$ and every λ such that $0<\lambda\leq 1/(1-\vartheta)r$ there holds the inequality

$$\varrho_1^\lambda(t,x)\leq\mu(\Omega)\varrho(t,\ \lambda(1-\vartheta)x)\quad \mu\text{-a.e. in } \Omega ,$$

then B maps $K_{\varrho_s}(r)$ into itself for every $\varkappa$ such that $0<|\varkappa|<1/\mu(\Omega)$.

Proof. It is sufficient to prove (a), but applying Jensen inequality, we get

$$\varrho(t,\lambda\varkappa\varrho(\cdot,x))\leq\frac{1}{\mu(\Omega)}\int_\Omega\left\{\int_\Omega k[t,s,\ \lambda k(s,u,\ |x(u)|)]\,d\mu(u)\right\}d\mu(s) =$$

$$=\frac{1}{\mu(\Omega)}\ \varrho_1^\lambda(t,x)\leq\ \varrho\left(t,\lambda\frac{R}{r}x\right),$$

and the assumptions of 3.1(b) are satisfied.

Proposition 5.2. (a) Let $0<|\varkappa|<1/\mu(\Omega)$ and let ϱ satisfy the condition from 5.1.(a) with R = r. Moreover, let us suppose that for every $\beta>0$ there exists $\gamma>0$ such that for all $x,y\in K_{\varrho_s}(r)$ there holds the inequality

$$\int_\Omega\left\{\frac{1}{\mu(\Omega)}\int_\Omega k\left[t,u,\ \frac{1}{\beta}|k(u,v,\ |x(v)|)-k(u,v,\ |y(v)|)|\right]d\mu(v)\right\}d\mu(u)\leq$$

(7)

$$\leq\int_\Omega k\left[t,u,\ \frac{|x(u)-y(u)|}{\gamma}\right]d\mu(u)\quad\text{for }\mu\text{-a.e. } t\in\Omega .$$

Then A maps $K_{\varrho_s}(r)$ into itself, continuously.

(b) Let $0 < |\varkappa| < 1/\mu(\Omega)$ and let ϱ satisfy the condition from 5.1.(b). Moreover, let us suppose that for every $\beta > 0$ there exists $\gamma > 0$ such that for all $x, y \in K_{\varrho_S}(r)$ there holds the inequality (7) for μ-a.e. $t \in \Omega$. Then B maps $K_{\varrho_S}(r)$ into itself, continuously.

Proof. We may limit ourselves to (a). Applying Jensen inequality and inequality (7), we obtain easily

$$\int_\Omega \varrho\left(t, \frac{I(\cdot,x) - I(\cdot,y)}{\eta\varepsilon}\right) d\mu(t) \leq$$

$$\leq \int_\Omega \left\{\int_\Omega k\left(t,u, \frac{|x(u)-y(u)|}{|\varkappa|\eta\delta}\right) d\mu(u)\right\} d\mu(t) = \int_\Omega \varrho\left(t, \frac{x-y}{\varkappa\eta\delta}\right) d\mu(t),$$

for μ-a.e. $t \in \Omega$, i.e. the inequality (4).

In a similar manner, the following statement may be proved applying 3.3.

Proposition 5.3. (a) Let $0 < |\varkappa| < 1/\mu(\Omega)$ and let ϱ satisfy the condition from 5.1.(a) with $R = r$. Moreover, let us suppose there exists a number $\alpha > 0$ such that for every $x, y \in K_{\varrho_S}(r)$ and for all $\eta > 0$ there holds the inequality

$$\int_\Omega \left\{\frac{1}{\mu(\Omega)} \int_\Omega k\left[t,u, \frac{\mu(\Omega)}{\eta} \left|k(u,v,|x(v)|) - k(u,v,|y(v)|)\right|\right] d\mu(v)\right\} d\mu(u) \leq \tag{8}$$

$$\leq \int_\Omega k\left[t,u, \frac{\alpha}{|\varkappa|\eta} |x(u) - y(u)|\right] d\mu(u) \quad \text{for } \mu\text{-a.e. } t \in \Omega .$$

Then $\|A(x) - A(y)\|_{\varrho_S} \leq \alpha\|x - y\|_{\varrho_S}$ for all $x, y \in K_{\varrho_S}(r)$. If $0 < \alpha < 1$, then A is a contraction operator in $K_{\varrho_S}(r)$.

(b) Let $0 < |\varkappa| < 1/\mu(\Omega)$ and let ϱ satisfy the condition from 5.1.(b). Moreover, let us suppose there exists a number $\alpha > 0$ such that for every $x, y \in K_{\varrho_S}(r)$ and for all $\eta > 0$ there holds the inequality (8). Then $\|B(x) - B(y)\|_{\varrho_S} \leq \alpha\|x - y\|_{\varrho_S}$ for all $x, y \in K_{\varrho_S}(r)$. If $0 < \alpha < 1$, then B is a contraction operator in $K_{\varrho_S}(r)$.

6. Applying Banach fixed-point theorem, the following result is deduced easily from Theorem 4.1 and Proposition 5.3.

Theorem 6.1. Let the kernel k satisfy the assumptions formulated in 1. Moreover, let us suppose that for every $u > 0$, the inequality

$\int_{\Omega} k(t,s,u)\,d\mu(t) > 0$ holds for μ-a.e. $s \in \Omega$. Let $0 < |\varkappa| < 1/\mu(\Omega)$, $0 < r < \infty$. Finally, we suppose that there exists a number α, $0<\alpha<1$, such that for every $x, y \in K_{\varrho_S}(r)$ and all $\eta > 0$ there holds the inequality (8) for μ-a.e. $t \in \Omega$. Then

(a) if $\varrho_1^{\lambda}(t,x) \leq \mu(\Omega)\, \varrho(t, \lambda x)$ μ-a.e. in Ω for every $x \in K_{\varrho_S}(r)$ and $0 < \lambda \leq 1/r$, then the integral equation (1) with $x_0(t) \equiv 0$ possesses only trivial solution in the ball $K_{\varrho_S}(r)$,

(b) if $\|x_0\|_{\varrho_S} \leq \vartheta r$, $0 < \vartheta < 1$, and $\varrho_1^{\lambda}(t,x) \leq \mu(\Omega) \varrho(t, \lambda(1-\vartheta)x)$ μ-a.e. in Ω for every $x \in K_{\varrho_S}(r)$ and $0 < \lambda \leq 1/(1-\vartheta)r$, then the integral equation (1) possesses exactly one solution in the ball $K_{\varrho_S}(r)$.

7. A special case of a kernel k is obtained if we take $k(t,s,u) = k_0(t,s)\,\varphi(u)$, where φ is a convex φ-function and $k_0 : \Omega \times \Omega \to [0,\infty)$ is a Σ-measurable, positive function in $\Omega \times \Omega$. By 4.1, X_{ϱ_S} is then a Banach space. Moreover, $\varrho_S(x) = \int_{\Omega} w(s)\,\varphi(|x(s)|)\,d\mu(s)$, where $w(s) = \int_{\Omega} k_0(t,s)\,d\mu(t) > 0$. Hence X_{ϱ_S} is an Orlicz space $L_w^{\varphi}(\Omega, \Sigma, \mu)$ with weight-function w, and $\|\ \|_{\varrho_S}$ is the norm in $L_w^{\varphi}(\Omega, \Sigma, \mu)$. Finally, we have then

$$k_1^{\lambda}(t,u,v) = |\lambda v| \int_{\Omega} k_0(t,s) k_0(s,u)\, d\mu(s),$$

$$\varrho_1^{\lambda}(t,x) = |\lambda| \int_{\Omega}\int_{\Omega} k_0(t,u) k_0(u,s)\, \varphi(|x(s)|)\, d\mu(u)\, d\mu(s).$$

Let us check the assumptions in case of the equation

$$(9) \qquad x(t) = \varkappa \int_0^t ts|x(s)|\,ds + x_0(t), \qquad 0 \leq t \leq 1.$$

Then $\varphi(u) = |u|$ and

$$k_0(t,s) = \begin{cases} ts & \text{for } 0 \leq s \leq t \\ 0 & \text{for } t < s \leq 1 \end{cases}, \qquad w(s) = \frac{1}{2}\, s(1-s^2),$$

$$\varrho(t,x) = \int_0^t ts|x(s)|\,ds, \qquad \varrho_S(x) = \frac{1}{2}\int_0^1 s(1-s^2)|x(s)|\,ds,$$

$$k_1^{\lambda}(t,u,v) = \frac{1}{3}|\lambda| tu(t^3-u^3)|v| \text{ for } 0 \leq u \leq t,\quad k_1^{\lambda}(t,u,v) = 0 \text{ for } t < u \leq 1,$$

$$\varrho_1^{\lambda}(t,x) = \frac{1}{3}|\lambda| \int_0^t ts(t^3-s^3)|x(s)|\,ds.$$

The inequality $\varrho_1^{\lambda}(t,x) \leq \varrho(t, \lambda(1-\vartheta)x)$ is satisfied for $0 < \vartheta < \frac{2}{3}$ and all $\lambda > 0$. The inequality (8) is satisfied, if only

$\frac{1}{3}\, tv\left(t^3 - v^3\right) \leq \frac{\alpha}{|\varkappa|}\, k_0(t,v)$, i.e. for $|\varkappa| \leq 3\alpha$. Hence, by Theorem 6.1, the equation (9) has exactly one solution in $K_{\varrho_s}(r)$, if $\| x_0 \|_{\varrho_s} \leq \vartheta r$, $0 < \vartheta < \frac{2}{3}$ and $0 < |\varkappa| < 1$.

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