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A HOMOGENEOUS HAUSDORFF E_0-SPACE WHICH IS NOT E_1

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In [3] C. E. Aull asks if there is a homogeneous Hausdorff space in which each point is a G_δ (an E_0-space) but is not the intersection of countably many closed neighborhoods (an E_1 space). In this note we shall construct such a space. An analogous construction will yield for each pair of infinite cardinals $\mathfrak{m} \leqq \mathfrak{n}$, a homogeneous Hausdorff space in which each point is the intersection of $\mathfrak{m}$ open sets but isn't the intersection of $\mathfrak{n}$ closed neighborhoods.

We shall first construct a space which satisfies the desired condition at one point (such spaces have been previously given by Novák [4, p. 89] and Aull [2, Example 1]; the basic idea behind all three spaces is the same) and then use copies of this space to construct a homogeneous space with the desired properties. (The basic idea here dates back to Urysohn [6, Kapitel III]; see also Shimrat [5] and the construction of S_ω in [1].)

For each countable ordinal α choose a sequence of points $\{x_n^\alpha\}$ (all of these points are distinct) and let $X_0 = (\omega_1 + 1) \cup \{x_n^\alpha \mid n \in N \text{ and } \alpha < \omega_1\}$ be topologized as follows: each x_n^α is an isolated point; a basic neighborhood of $\alpha < \omega_1$ is of the form $\{\alpha\} \cup \{x_n^\alpha \mid n \geqq k\}$; a basic neighborhood of ω_1 is of the form

$$V(\beta, k) = \{x_n^\alpha \mid n \geqq k,\ \alpha \geqq \beta\} \cup \{\omega_1\}.$$

Clearly X_0 is a Hausdorff (in fact a Urysohn) space. It is not regular since the set ω_1 cannot be separated from the point ω_1. Each point is a G_δ since $\{\omega_1\} = \bigcap_{k=1}^{\infty} V(0, k)$, but ω_1 is not the intersection of any countable collection of it's closed neighborhoods.

Let us now modify X_0 by adding a sequence of points converging to ω_1 as follows: Let $X = X_0 \cup \{x_n\}$ (where $\{x_n\} \cap X_0 = \emptyset$) with each $V(\beta, k) \cup \{x_n \mid n \geqq m\}$ a basic neighborhood of ω_1 in X and each x_n isolated. Clearly X has all the properties asserted for X_0 in the last paragraph.

We shall now construct a sequence $\{H_n\}$ of spaces, each a subspace of the succeeding ones, whose union (i.e. inductive limit) will be the desired space. Let $H_1 = X$. To construct H_2 from H_1, attach to each point x_n^α (x_n) of H_1 a copy X_n^α (X_n) of X by identifying x_n^α with $(\omega_1)_n^\alpha \in X_n^\alpha$, and similarly for x_n, and to each $\alpha \in H_1$ $(\alpha < \omega_1)$ a copy

X_0^α of X_0 by identifying α with $(\omega_1)^\alpha \in X_0^\alpha$. The resulting space H_2 should be given the quotient topology. In general, to construct H_n from H_{n-1}, attach to each isolated point x of H_{n-1} a copy X^x of X by identifying x with $(\omega_1)^x \in X$, and to each non-isolated point $y \in H_{n-1}$ with a countable basis of neighborhoods a copy X_0^y of X_0 identifying y and $(\omega_1)^y \in X_0^y$. Again the resulting H_n should have the quotient topology. Let H be the inductive limit of the H_n.

It is easy to verify that H is a Urysohn space and that the point of H arising from $\omega_1 \in H_1$ is a G_δ that isn't the intersection of countably many closed neighborhoods. It remains only to show that H is homogeneous. To this end, we partially order H as follows: each x (or y) in H_{n-1} is greater than ($>$) each point of X^x (or X_0^y) and y is greater than each point of its basic neighborhoods in H_{n-1}; the desired relation is the transitive closure of this one. Now for any $x \in H$, $L(x) = \{y \in H \mid y \leqq x\}$ is a closed subspace of H that is homeomorphic to H. If $H \setminus L(x) \neq \emptyset$, it is also homeomorphic to H. Hence for any $x \neq \omega_1 \in H$, there is a homeomorphism $h_x : H \to H$ interchanging x and ω_1. Hence H is homogeneous.

The construction is easily modified to take care of cardinals $\mathfrak{m} \leqq \mathfrak{n}$, producing a homogeneous Urysohn space in which each point is the intersection of $\mathfrak{m}$ open sets but not the intersection of any $\mathfrak{n}$ of its closed neighborhoods.

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