Inara Yermachenko

Multiple solutions of nonlinear boundary value problems by the quasilinearization process

In: Marek Fila and Angela Handlovičová and Karol Mikula and Milan Medveď and Pavol Quittner and Daniel Ševčovič (eds.): Proceedings of Equadiff 11, International Conference on Differential Equations. Czecho-Slovak series, Bratislava, July 25-29, 2005, [Part 2] Minisymposia and contributed talks. Comenius University Press, Bratislava, 2007. Presented in electronic form on the Internet. pp. 577--587.

Persistent URL: http://dml.cz/dmlcz/700458

Proceedings of Equadiff-11
2005, pp. 577–587
ISBN 978-80-227-2624-5

MULTIPLE SOLUTIONS OF NONLINEAR BOUNDARY VALUE PROBLEMS BY THE QUASILINEARIZATION PROCESS*

INARA YERMACHENKO†

Abstract. We investigate second and fourth order boundary value problems (BVPs) for the Emden-Fowler type equations using the quasilinearization process. We reduce the given nonlinear equation to a some quasi-linear one with a non-resonant linear part so that both equations are equivalent in some bounded domain. We show that modified problem has a solution of the same oscillatory type as the linear part has. We prove that under certain conditions quasilinearization process can be applied with essentially different linear parts and hence the original problem is shown to have multiple solutions.

Key words. Quasi-linear equation, quasilinearization, non-resonant linear part, type of a solution

AMS subject classifications. 34B15

1. Introduction.

Consider the nonlinear differential equation

$$x^{(n)} = f(t,\, x), \tag{1.1}$$

where $n = 2,\, 4, \quad t \in I := [0, 1]$ with some boundary conditions

$$U_\mu(x) := \sum_{\nu=0}^{n-1} [\alpha_\mu^{(\nu)} x^{(\nu)}(0) + \beta_\mu^{(\nu)} x^{(\nu)}(1)] = \gamma_\mu, \quad (\mu = 1, \ldots, n). \tag{1.2}$$

Function $f : I \times \mathbb{R} \to \mathbb{R}$ is supposed to be continuous. We investigate existence and multiplicity of solutions to the BVP (1.1), (1.2) in some specific cases by reducing it to multiple quasi-linear problems of different types.

Suppose that equation (1.1) can be reduced to the quasi-linear one of the form

$$\big(L_n x\big)(t) = F_1(t,\, x), \tag{1.3}$$

where F_1 is continuous and bounded, that is, there exists $M_1 \in (0, +\infty)$ such that $\big|F_1(t,\, x)\big| < M_1$ for any values of arguments, and $\big(L_n x\big)(t)$ is a non-resonant linear part with respect to the given boundary conditions (1.2), that is, the homogeneous problem $\big(L_n x\big)(t) = 0$, $U_\mu(x) = 0$ has only the trivial solution. Then the modified quasi-linear problem (1.3), (1.2) is solvable.

Suppose also that equations (1.1), (1.3) are equivalent in some domain $\Omega_1(t, x)$. If a solution $x_1(t)$ of the quasi-linear problem (1.3), (1.2) is located in the domain of equivalence $\Omega_1(t, x)$, then this $x_1(t)$ also solves the original problem (1.1), (1.2).

If the equation (1.1) can be reduced to another quasi-linear equation

$$\big(l_n x\big)(t) = F_2(t,\, x), \tag{1.4}$$

which is equivalent to (1.1) in the domain $\Omega_2(t, x)$, then the original problem (1.1), (1.2) in some cases has another solution $x_2(t)$ $\Big(x_2(t) \in \Omega_2(t, x)\Big)$.

*This work was supported by the project VPD1/ESF/PIAA/04/NP/3.2.3.1/0003/0065.
†Department of Mathematics and Natural Sciences, Daugavpils University, Parades str. 1, Daugavpils, Latvia, LV-5400 (inari@dau.lv).

If the linear parts $(L_n x)(t)$ and $(l_n x)(t)$ are essentially different, then we can prove, that there exist $x_L(t)$ and $x_l(t)$, which are the solutions of different types. In this way one can obtain the multiplicity results.

Our research is motivated by the papers of R. Conti [1], L. Erbe [2], L. Jackson and K. Schrader [3], who studied oscillatory properties of solutions of two-point second-order boundary value problems. To obtain the results for the fourth-order BVPs we use the oscillation theory by Leighton-Nehari [5] for the fourth-order linear differential equations.

2. Second-order boundary value problems.

2.1. Second-order quasi-linear problems and types of solutions. Consider the quasi-linear problem

$$\big(L_2 x\big)(t) = F(t, x), \quad x(0) = x(1) = 0, \tag{2.1}$$

where $F \in C([0, 1] \times \mathbb{R}, \mathbb{R})$ and $\big|F(t, x)\big| < M \quad \forall (t, x) \in [0, 1] \times \mathbb{R}$. Several definitions will be used in the sequel.

Definition 2.1. We will say that the linear part $\big(L_2 x\big)(t)$ is i-nonresonant in the interval $[0, 1]$ with respect to the Dirichlet boundary conditions $x(0) = 0, \quad x(1) = 0$ if a solution of the Cauchy problem

$$\big(L_2 x\big)(t) = 0, \quad x(0) = 0, \quad x'(0) = 1 \tag{2.2}$$

has exactly i zeros in the interval $(0, 1)$ and $x(1) \neq 0$.

If in the interval $[0, 1]$ a linear part $(L_2 x)(t)$ is i-nonresonant, but a linear part $(l_2 x)(t)$ is j-nonresonant and $i \neq j$, then we will say that the linear parts $(L_2 x)(t)$ and $(l_2 x)(t)$ are essentially different.

For instance, the linear part $\big(L_2 x\big)(t) := x'' + k^2 x$ is non-resonant with respect to the Dirichlet boundary conditions $x(0) = 0, \quad x(1) = 0$ (it means that the respective homogeneous problem has only a trivial solution), if coefficient k belongs to one of the intervals

$$(0, \pi),\ (\pi, 2\pi), \ldots, (i\pi, (i+1)\pi), \ldots$$

For the values of k from different intervals the solutions of the Cauchy problems

$$x'' + k^2 x = 0, \quad x(0) = 0, \quad x'(0) = 1 \tag{2.3}$$

with different oscillatory properties are obtained and shown below.

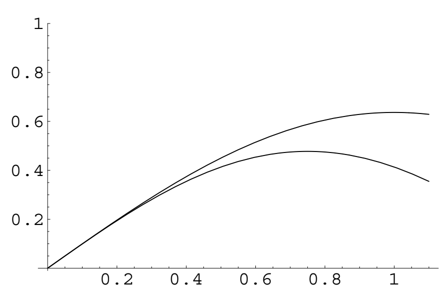

Fig. 2.1. 0-*nonresonance,* $k \in (0, \pi)$.

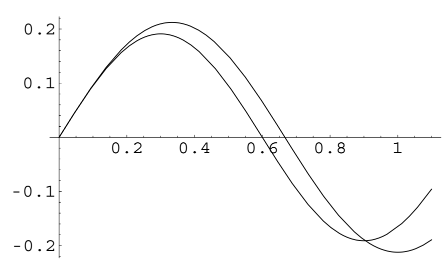

Fig. 2.2. 1-*nonresonance,* $k \in (\pi, 2\pi)$.

FIG. 2.1 shows the solutions of the problems (2.3) for $k_1 = \frac{\pi}{2}$ and $k_2 = \frac{2\pi}{3}$. The linear parts $x'' + k_1^2 x$ and $x'' + k_2^2 x$ are 0-nonresonant (see DEFINITION 2.1).

FIG. 2.2 illustrates the solutions of the problems (2.3) for $k_3 = \frac{3\pi}{2}$ and $k_4 = \frac{5\pi}{3}$. The respective linear parts $x'' + k_3^2 x$ and $x'' + k_4^2 x$ are 1-nonresonant. Therefore the linear parts $x'' + (\frac{\pi}{2})^2 x$ and $x'' + (\frac{3\pi}{2})^2 x$ are essentially different.

We remark that a linear part $\big(L_2 x\big)(t) := x'' + k^2 x$ is n-nonresonant for $k = \dfrac{\pi}{2} + \pi n$, $n = 0,\, 1,\, 2, \ldots$; the linear parts $\big(L_2 x\big)(t) := x'' + k^2 x$ for different values of k in the form $\dfrac{\pi}{2} + \pi n$ are essentially different.

DEFINITION 2.2. We will say that $\xi(t)$ is an i-type solution of the quasi-linear problem (2.1) if for small enough $\alpha > 0$ the difference $u(t;\alpha) = x(t;\alpha) - \xi(t)$ has exactly i zeros in the interval $(0, 1)$ and $u(1;\alpha) \neq 0$, where $x(t;\alpha)$ is a solution of the quasi-linear equation

$$\big(L_2 x\big)(t) = F(t,\, x), \tag{2.4}$$

which satisfies the initial conditions

$$x(0;\alpha) = \xi(0) = 0, \qquad x'(0;\alpha) = \xi'(0) + \alpha. \tag{2.5}$$

In what follows we call the solution $x(t;\alpha)$ by *neighboring* solution.

REMARK . An i-type solution $\xi(t)$ of problem (2.1) has the following characteristics in terms of the variational equation: a solution $y(t)$ of the variational equation $\big(L_2 y\big)(t) = F_x(t,\, \xi(t))y$, which satisfies the initial conditions $y(0) = 0, \quad y'(0) = 1$, either has exactly i zeros in the interval $(0,\, 1)$ and $y(1) \neq 0$, or it has i-th zero or $(i+1)$-th zero at $t = 1$. (The corresponding examples for the cases of i-th or $(i+1)$-th zero of $y(t)$ being at $t = 1$ can be constructed). Therefore it may happen that $y(t)$, corresponding to a solution $\xi(t)$ of i-type, has its $(i+1)$-th zero at $t = 1$, and $y(t)$, corresponding to a solution $\xi(t)$ of $(i+1)$-type, also has $(i+1)$-th zero at $t = 1$.

The following theorem is valid [7].

THEOREM 2.3. *Quasi-linear problem* (2.1) *with an i-nonresonant linear part* $\big(L_2 x\big)(t)$ *has an i-type solution.*

Consider a nonlinear second-order boundary value problem

$$x'' = f(t,\, x), \qquad x(0) = x(1) = 0. \tag{2.6}$$

Function f is supposed to be continuous.

DEFINITION 2.4. Let the differential equation in (2.6) and quasi-linear equation (2.4) be equivalent in a domain

$$\Omega_N = \{(t, x) :\ 0 \le t \le 1,\ |x| \le N\}. \tag{2.7}$$

Suppose that any solution $x(t)$ of the quasi-linear problem (2.1) satisfies the estimate $\big|x(t)\big| < N, \quad \forall t \in [0,\, 1]$. We will then say that problem (2.6) allows for quasilinearization with respect to a linear part $\big(L_2 x\big)(t)$ and a domain Ω_N.

The propositions below follow from THEOREM 2.3.

PROPOSITION 2.5. *If the problem* (2.6) *allows for quasilinearization with respect to some domain Ω_N and some i-nonresonant linear part* $\big(L_2 x\big)(t)$, *then it has an i-type solution.*

Proposition 2.6. *If the problem* (2.6) *allows for quasilinearization with respect to* n *domains of the form* (2.7) *and* n *essentially different (in the sense of* Definition 2.1*) linear parts, then it has at least* n *solutions of different types.*

Analogous results on multiple solutions were obtained for the second-order Neumann [8] and Sturm–Liouville [9] boundary value problems.

2.2. Second-order nonlinear BVPs. In this section we show by using the quasilinearization process how multiple solutions of different types can be obtained for the Dirichlet boundary value problem to the Emden – Fowler type equation.

Consider the second-order boundary value problem

$$x'' = -\lambda^2 |x|^p \operatorname{sign} x, \qquad x(0) = x(1) = 0, \tag{2.8}$$

where $p > 0, \quad p \neq 1, \lambda \neq 0.$

Theorem 2.7. *If the inequality*

$$\frac{k}{|\sin k|} < \beta \frac{p^{\frac{p}{p-1}}}{|p-1|}, \tag{2.9}$$

where β *is the positive root of the equation* $\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}$, *holds for some value* $k \in (i\pi, (i+1)\pi)$, $(i = 0, 1, \ldots)$, *then there exists an* i*-type solution of the problem* (2.8).

Proof. The equation in (2.8) can be reduced to the equivalent equation

$$x'' + k^2 x = k^2 x - \lambda^2 |x|^p \operatorname{sign} x, \tag{2.10}$$

where k satisfies $i\pi < k < (i+1)\pi$ for some $i = 0, 1, \ldots$.

Denote the right side of the equation (2.10) by $f_k(x)$ and try to bound it by a number M_k, which is an absolute value of the function $f_k(x)$ at the point of extremum. We can calculate this number M_k

$$M_k = |f_k(x_0)| = \lambda^{\frac{2}{1-p}} \cdot \left(\frac{k^2}{p}\right)^{\frac{p}{p-1}} \cdot |p-1| \tag{2.11}$$

and corresponding number N_k such, that if $|x(t)| \le N_k$, then $|f_k(x)| \le M_k$.

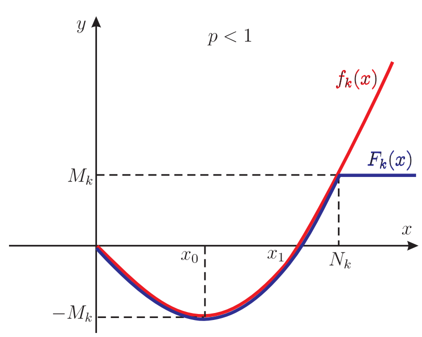

Fig. 2.3. *Existence of a number* N_k

Computation gives that

$$N_k = \left(\frac{k^2}{\lambda^2}\right)^{\frac{1}{p-1}} \beta, \tag{2.12}$$

where β is a positive root of the equation $\beta^p = \beta + (p-1)\cdot p^{\frac{p}{1-p}}$. Then one can consider the quasi-linear equation

$$x'' + k^2 x = F_k(x), \tag{2.13}$$

where $F_k(x) := f_k\Big(\delta\big(-N_k,\, x,\, N_k\big)\Big)$ and δ is a "cut-off" function, that is,

$$\delta\big(-N_k,\, x,\, N_k\big) = \begin{cases} N_k, & x > N_k, \\ x, & -N_k \le x \le N_k, \\ -N_k, & x < -N_k \end{cases}$$

and $\max\{|F_k| : x \in R\} = M_k$.

The given Emden-Fowler equation in (2.8) is equivalent to the equation (2.13) in the domain $\Omega_k = \{(t,\, x) : 0 \le t \le 1, \quad |x(t)| \le N_k\}$. The obtained quasi-liner problem ((2.13) with given Dirichlet boundary conditions) can be rewritten in integral form $x(t) = \int\limits_0^1 G_k(t,s) F_k(x(s))\, \mathrm{d}s$, where $G_k(t,s)$ is the Green's function of the respective homogeneous problem. It is given by

$$G_k(t,s) = \begin{cases} \dfrac{\sin k(s-1)\cdot \sin kt}{k \sin k}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{\sin k(t-1)\cdot \sin ks}{k \sin k}, & 0 \le s < t \le 1 \end{cases} \tag{2.14}$$

and satisfies the estimate

$$|G_k(t,s)| \le \Gamma_k := \frac{1}{k\cdot|\sin k|}. \tag{2.15}$$

Then

$$\big|x(t)\big| \le \Gamma_k \cdot M_k. \tag{2.16}$$

If the inequality

$$\Gamma_k \cdot M_k < N_k \tag{2.17}$$

holds, then a solution $x(t)$ of the quasi-linear problem satisfies the estimate $\big|x(t)\big| < N_k$, $\forall t \in [0,\, 1]$ and the original problem (2.8) allows for quasilinearization with respect to the domain Ω_k and the linear part $\big(L_2 x\big)(t) := x'' + k^2 x$. It follows from Proposition 2.5 that the problem (2.8) has an i-type solution. It follows from (2.11), (2.12), (2.15) that the inequality (2.17) reduces to (2.9). □

Corollary 2.8. *If there exist numbers* $k_j \in (j\pi,\, (j+1)\pi)$, $(j = 0,\, 1,\, \ldots, n)$, *which satisfy the inequality* (2.9), *then there exist at least* $n+1$ *solutions of different types to the problem* (2.8).

We have obtained the results, which show that certain values of k in the form $\dfrac{\pi}{2} + \pi n$, $n = 0,\, 1,\, 2, \ldots$ for some values of p are good for the inequality (2.9) to be satisfied [10]. For instance, if $p = \frac{8}{7}$, then there exist at least three values $k_0 = \frac{\pi}{2}$, $k_1 = \frac{3\pi}{2}$, $k_2 = \frac{5\pi}{2}$,which satisfy the inequality (2.9).

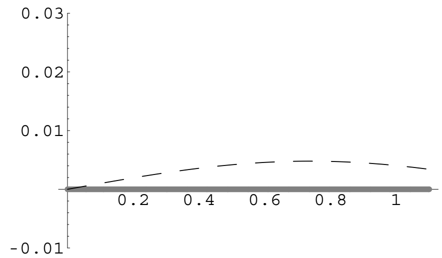

FIG. 2.4. 0-*type solution*

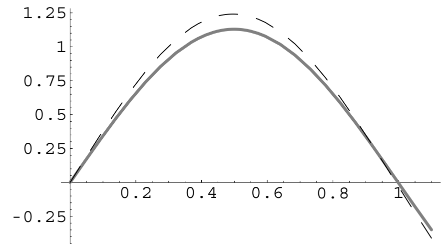

FIG. 2.5. 1-*type solution*

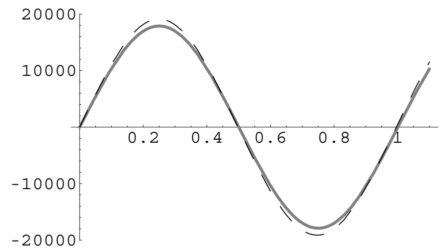

FIG. 2.6. 2-*type solution*

We can show the solutions of different types, for example, considering the problem

$$x'' = -10\,|x|^{\frac{8}{7}}\,\text{sign}\; x, \quad x(0) = x(1) = 0. \tag{2.18}$$

The solid line in the FIGURES 2.4, 2.5, 2.6 indicates respectively 0-type, 1-type and 2-type solution of the problem (2.18) and the dashed line relates to one of the corresponding neighboring solutions (see DEFINITION 2.2).

3. Fourth-order boundary value problems. We have obtained the similar multiplicity results for some the fourth-order nonlinear BVPs.

3.1. Fourth-order quasi-linear problems and types of solutions. Consider two-point boundary value problem

$$x^{(4)} = f(t,\, x), \tag{3.1}$$

$$x(0) = x'(0) = 0 = x(1) = x'(1), \tag{3.2}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$ is continuously differentiable. We first prove results for quasi-linear problems of the type

$$x^{(4)} - k^4 x = F(t, x), \tag{3.3}$$

(3.2), where $t \in I := [0, 1]$, F, $F_x : I \times \mathbb{R} \to \mathbb{R}$ are continuous, F is bounded and the following condition is satisfied for any $(t,\, x)$

$$k^4 + \frac{\partial F}{\partial x}(t,\, x) > 0. \tag{3.4}$$

In our investigation we use the oscillation theory by Leighton-Nehari for the fourth-order linear differential equations [5]

$$x^{(4)} - p(t)x = 0, \qquad p(t) > 0. \tag{3.5}$$

We used their definition of a conjugate point.

Definition 3.1. A point η is called by *a conjugate point* for the point $t = 0$, if there exists a nontrivial solution $x(t)$ such that $x(0) = x'(0) = 0 = x(\eta) = x'(\eta)$.

For example, if the linear equation is $x^{(4)} - k^4 x = 0$, then the conjugate points η satisfy $\cos k\eta \cdot \cosh k\eta = 1$.

The conjugate points (or double zeros) in the oscillation theory for the fourth-order linear differential equations play the same role as the ordinary zeros in the oscillation theory for the second-order equations.

We define i-nonresonanse of the linear part and an i-type solution similarly as for the second-order quasi-linear problems.

Definition 3.2. We will say that the linear part $(L_4x)(t) := x^{(4)} - k^4 x$ is i-nonresonant with respect to the boundary conditions (3.2), if there are exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point.

For example, the linear part $\big(L_4x\big)(t) := x^{(4)} - k^4 x$ is $(n-1)$-nonresonant for $k = \pi n$, $n = 1, 2, \ldots$.

Definition 3.3. We will say that $\xi(t)$ is an i-type solution of the problem (3.3), (3.2), if for small enough $\alpha, \beta > 0$ the difference $u(t; \alpha, \beta) = x(t; \alpha, \beta) - \xi(t)$ has exactly i double zeros (for different values of α, β) in the interval $(0, 1)$ and $u(1; \alpha, \beta) \neq 0$, where $x(t; \alpha, \beta)$ is a solution of (3.3), which satisfies the initial conditions

$$x(0; \alpha, \beta) = \xi(0), \quad x'(0; \alpha, \beta) = \xi'(0), \tag{3.6}$$

$$x''(0; \alpha, \beta) = \xi''(0) + \alpha, \quad x'''(0; \alpha, \beta) = \xi'''(0) - \beta. \tag{3.7}$$

We call the solution $x(t; \alpha, \beta)$ by *neighboring* solution.

Remark. An i-type solution $\xi(t)$ of the problem (3.3), (3.2) has the following characteristics in terms of the variational equation: if a linear equation of variations $y^{(4)} - k^4 y = F_x(t, \xi(t))y$ has exactly i conjugate points in the interval $(0, 1)$ and $t = 1$ is not a conjugate point, then $\xi(t)$ is an i-type solution. However, if $t = 1$ is a conjugate point, then $\xi(t)$ may be an i-type solution, or it may be an $(i + 1)$-type solution, or its type may be indefinite. The respective examples can be constructed.

The following theorem is valid.

Theorem 3.4. *The quasi-linear problem* (3.3), (3.2) *has an i-type solution, if the condition* (3.4) *is fulfilled and the linear part $(L_4x)(t) = x^{(4)} - k^4 x$ is i-nonresonant.*

Theorem 3.4 was proved in [11], [6].

3.2. Green's function for the oscillatory fourth-order linear problem. Consider the problem

$$\begin{cases} x^{(4)} - k^4 x = 0, \\ x(0) = x'(0) = 0 = x(1) = x'(1), \end{cases} \tag{3.8}$$

where k satisfies the non-resonance condition $\cos k \cosh k \neq 1$. We have constructed the Green's function for the problem (3.8) and give the respective formula and its estimate below.

PROPOSITION 3.5. *The Green's function of the problem* (3.8) *can be written in the form*

$$G_k(t,\, s) = \begin{cases} \frac{1}{\triangle}\Big(-u^*(t,\, s)\cdot v(1) - u(1)\cdot v^*(t,\, s) + \sum\limits_{\tau=s,\, t} \big[u(\tau)\cdot v(t+s-\tau) \\ \qquad -u(\tau-1)\cdot v(t+s-1-\tau) - u(t-\tau)\cdot v(\tau-s)\big]\Big), \\ \qquad 0 \le s \le t \le 1, \\ \frac{1}{\triangle}\Big(-u^*(s,\, t)\cdot v(1) - u(1)\cdot v^*(s,\, t) + \sum\limits_{\tau=s,\, t} \big[u(\tau)\cdot v(t+s-\tau) \\ \qquad -u(\tau-1)\cdot v(t+s-1-\tau) + u(t-\tau)\cdot v(\tau-s)\big]\Big), \\ \qquad 0 \le t < s \le 1 \end{cases} \tag{3.9}$$

where $\triangle = 4k^3(\cos k \cosh k - 1)$ *and* $u,\, v$ *are vector-functions such that*
$u(\tau) = [-\sin k\tau,\, \cos k\tau]$, $u^*(t,\, s) = [-\sin k(s-t+1),\, \cos k(t+s-1)]$,
$v(\tau) = [\cosh k\tau,\, \sinh k\tau]$, $v^*(t,\, s) = [\cosh\ k(t+s-1),\, \sinh k(s-t+1)]$, *and the dot* $u\cdot v$ *denotes the scalar product.*

Proof. The Green's function is constructed as an element of the Green's matrix by reducing the linear problem (3.8) to a matrix form [4]. □

PROPOSITION 3.6. *The Green's function* $G_k(t,s)$ (3.9) *can be estimated by*

$$\big|G_k(t,s)\big| \le \Gamma_k := \frac{(5+\sqrt{2})\sqrt{\cosh 2k} + \sinh k + 1}{4k^3\big|\cos k\cdot\cosh k - 1\big|}. \tag{3.10}$$

Proof. Follows from a property of the scalar product $|u\cdot v| \le |u|\cdot|v|$ taking into consideration that $|u(\tau)| \le 1$, $|v^*(t,s)| \le \cosh 2k$, $|u^*(t,s)| \le \sqrt{2}$, $|v^*(t,s)| \le \sqrt{\cosh 2k}$. □

We can improve this estimate for some numbers k. For instance, if $k = \pi n$, $(n = 1, 2, \ldots)$ the Green's function $G_k(t,\, s)$ can be simplified. We express hyperbolic sine and cosine in terms of the exponential functions and obtain the following estimates

$$\big|G_k(t,s)\big| \le \frac{(1+\sqrt{2})\,\mathrm{e}^k}{k^3(\mathrm{e}^k+1)} =: \Gamma_1(k), \qquad \text{if} \quad k = (2n-1)\pi, \tag{3.11}$$

$$\big|G_k(t,s)\big| \le \frac{(1+\sqrt{2})\,\mathrm{e}^k}{k^3(\mathrm{e}^k-1)} =: \Gamma_2(k), \qquad \text{if} \quad k = 2n\pi. \tag{3.12}$$

3.3. Emden-Fowler equation. We apply the obtained estimates (3.11), (3.12) and THEOREM 3.4 to the Emden-Fowler type equation

$$x^{(4)} = \lambda^2\cdot|x|^p\,\mathrm{sign}\, x, \tag{3.13}$$

where $\lambda \neq 0$, $p > 0$, $p \neq 1$, with the boundary conditions (3.2).

THEOREM 3.7. *If there exists some* k *in the form* $k = \pi i$, $(i = 1, 2, \ldots)$, *which satisfies the inequality*

$$k\cdot\frac{(1+\sqrt{2})e^k}{(e^k+1)} < \beta\cdot\frac{p^{\frac{p}{p-1}}}{|p-1|} \qquad \textit{for} \quad k = (2n-1)\pi \tag{3.14}$$

or

$$k \cdot \frac{(1+\sqrt{2})e^k}{(e^k-1)} < \beta \cdot \frac{p^{\frac{p}{p-1}}}{|p-1|} \qquad \textit{for} \quad k = 2n\pi, \tag{3.15}$$

where β *is a positive root of the equation* $\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}$, *then there exists an* $(i-1)$*-type solution of the problem* (3.13), (3.2).

Proof. Let us consider instead of the equation (3.13) the equivalent one

$$x^{(4)} - k^4 x = \lambda^2 \cdot |x|^p \operatorname{sign} x - k^4 x, \tag{3.16}$$

where k satisfies $\cos k \cosh k \neq 1$. Denote $f_k(x) := \lambda^2 \cdot |x|^p \operatorname{sign} x - k^4 x$. We can calculate the value of the function $f_k(x)$ at the point of extremum x_{extr}

$$M_k = |f_k(x_{extr})| = \lambda^{\frac{2}{1-p}} \cdot \left(\frac{k^4}{p}\right)^{\frac{p}{p-1}} \cdot |p-1|. \tag{3.17}$$

Choose $N_k > 0$ such that $|x(t)| \leq N_k \Rightarrow |f_k(x)| \leq M_k, \quad \forall t \in I$. Computation gives that

$$N_k = \left(\frac{k^4}{\lambda^2}\right)^{\frac{1}{p-1}} \beta, \tag{3.18}$$

where β is a positive root of the equation $\beta^p = \beta + (p-1) \cdot p^{\frac{p}{1-p}}$.

The rest proof is similar to the proof of Theorem 2.7 for the second-order problems.

Consider the quasi-linear equation

$$x^{(4)} - k^4 x = F_k(x), \tag{3.19}$$

where $F_k(x) = f_k(x) \ \forall x \in \{x : |x(t)| \leq N_k\}$ and rewrite the obtained quasi-linear problem (3.19), (3.2) in the integral form. Then a solution $x(t)$ of the problem (3.19), (3.2) satisfies the estimate $\big|x(t)\big| \leq \Gamma_k \cdot M_k$, where Γ_k the estimate of the Green's function (3.9). If moreover the inequality

$$\Gamma_k \cdot M_k \; < \; N_k \tag{3.20}$$

holds, then equations (3.13) and (3.19) are equivalent in the domain $\Omega_k = \{(t,x) : \ 0 \leq t \leq 1, \ |x| < N_k\}$. In other words if the inequality (3.20) holds, then the original problem (3.13), (3.2) allows for quasilinearization with respect to the domain Ω_k and the linear part $(L_4x)(t) = x^{(4)} - k^4 x$.

Notice that in this domain of equivalence Ω_k the condition (3.4) is fulfilled (i.e. $k^4 + \dfrac{dF_k}{dx} > 0$). So it follows from Theorem 3.4 that if the linear part $(L_4x)(t) = x^{(4)} - k^4 x$ is i-nonresonant, then the quasi-linear problem (3.19), (3.2) has an i-type solution, if moreover the inequality (3.20) holds, then the original problem (3.13), (3.2) also has an i-type solution.

Let us consider values k of the form $k = \pi i \, (i = 1, 2 \ldots)$. For such k the linear part $(L_4x)(t) = x^{(4)} - k^4 x$ is $(i-1)$-nonresonant and the Green's function $G_k(t,s)$ satisfies either the estimate $\Gamma_1(k)$ (3.11) or $\Gamma_2(k)$ (3.12). It follows from (3.17), (3.18), (3.11), (3.12) that the inequality (3.20) reduces respectively either to (3.14) or (3.15). ∎

Corollary 3.8. *If there exist* $k = \pi i, \ i = 1, 2, \ldots, m$, *which satisfy the inequalities* (3.14), (3.15), *then there exist at least* m *solutions of different types to the problem* (3.13), (3.2).

We have obtained the results, which show that certain values of k in the form πi, $i = 1, 2, \ldots$ for some values of p are good for the inequalities (3.14) or/and (3.15) to be satisfied [11].

We illustrate our considerations by an example. Consider the problem

$$\begin{aligned} x^{(4)} &= 810 \cdot |x|^{\frac{8}{7}} \operatorname{sign} x, \\ x(0) = x'(0) &= 0 = x(1) = x'(1). \end{aligned} \tag{3.21}$$

We have computed the solutions of different types for this problem.

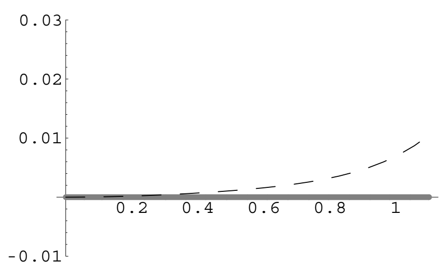

Fig. 3.1. 0-*type solution*

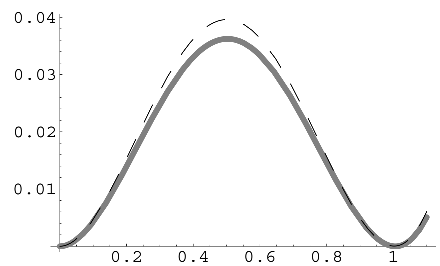

Fig. 3.2. 1-*type solution*

The solid line in Figure 3.1 indicates a trivial solution of the problem (3.21) and the dashed line relates to one of the corresponding neighboring solutions (see Definition 3.3). All other neighboring solutions are such that the difference has no double zero in the interval (0, 1), therefore the trivial solution is a 0-type solution.

Figure 3.2 shows another solution of the problem (3.21) in solid and it is an 1-type solution; the difference between neighboring solution (dashed line) and this solution has exactly one double zero (conjugate point) in some point of the open interval (0, 1). The initial data of the 1-type solution are $x''(0) = 1.1, \quad x'''(0) = -5.03461937$.

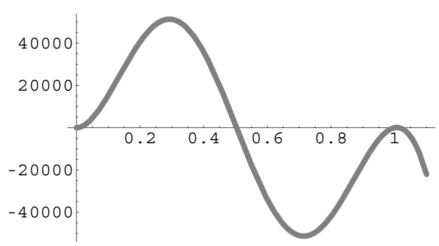

Fig. 3.3. 2-*type solution*

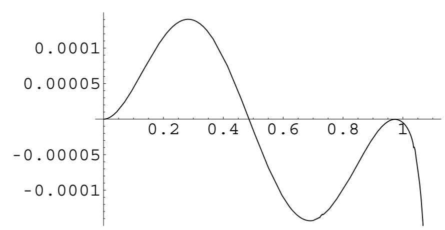

Fig. 3.4. *Difference between* 2*type solution and respective neighboring solution*

Figure 3.3 illustrates a 2-type solution of the problem (3.21). The graph of the respective neighboring solution is difficult to show, because two lines almost coincide. Nevertheless, the difference between neighboring solution and this solution is depicted in Figure 3.4 and it has one simple zero and one double zero in the open interval (0, 1). Then another neighboring solution $x(t;\alpha,\beta)$ exists such that the difference $x(t;\alpha,\beta) - \xi(t)$ has only one zero in (0, 1) and this zero is a double zero. The initial data of the 2-type solution are large comparing with those for previous solutions, $x''(0) = 4099959.008, \quad x'''(0) = -31634999.21$.

Acknowledgments. This work was advised by prof. F. Sadyrbaev (Institute of Mathematics and Computer Science, University of Latvia).

REFERENCES

[1] R. Conti, *Equazioni differenziali ordinarie quasilineari con condizioni lineari*, Ann. mat. pura ed appl., **57** (1962), 49.

[2] L. H. Erbe, *Nonlinear boundary value problems for second order differential equations*, J. Differential equations, **7** (1970), 459–472.

[3] L. K. Jackson and K. W. Schrader, *Comparison theorems for nonlinear differential equations*, J.Differential equations, **3** (1967), 258–255.

[4] I. Kiguradze, *Initial and boundary value problems for systems of ordinary differential equations (Vol.1 Linear theory)*, Metsniereba-Publishing House, Tbilisi, 1997, (in Russian).

[5] W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc., **89** (1961), 325–377.

[6] F. Sadyrbaev and I. Yermachenko, *Types of solutions and multiplicity results for the fourth-order boundary value problems*, Proc. of the Conference on Differential and Difference Equations, Melbourne, Fl., USA, August 2005, to appear.

[7] I. Yermachenko and F. Sadyrbaev, *Types of solutions and multiplicity results for two-point nonlinear boundary value problems*, Elsevier Ltd., Nonlinear Analysis **63** (2005), e1725–e1735. (electr. version `http://www.elsevier.com/locate/na`)

[8] I. Yermachenko and F. Sadyrbaev, *Types of solutions of the second order Neumann problem: multiple solutions*, Mathematics. Differential equations (Univ. of Latvia, Institute of Math. and Comp. Sci.), **4** (2004), 5–21.

[9] I. Yermachenko, *Multiple solutions of Sturm-Liouville type boundary value problems*, Proceedings of the 5th Latvian Math. Conference, Daugavpils, Latvia, 2004, Acta Societatis Math. Latviensis, **6** (2004), 61.

[10] I. Yermachenko and F. Sadyrbaev, *Quasilinearization and multiple solutions of the Emden-Fowler type equation*, Math. Modelling and Analysis, **10**(1) (2005), 41–50.

[11] I. Yermachenko and F. Sadyrbaev, *Types of solutions and multiplicity results for two-point fourth-order nonlinear BVP*, Mathematics. Differential equations (Univ. of Latvia,Institute of Math. and Comp. Sci.), **5** (2005), 33–46.