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A LIMITING CASE OF THE UNCERTAINTY PRINCIPLE

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Abstract. We prove an imbedding inequality in the form of the uncertainty principle, independent of the dimension.

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1. Introduction, notations. Our concern in this paper lies with the weighted inequality

$$\left(\int_B f(x)^2 V(x)\ \mathrm{d}x\right)^{1/2} \leq c\left(\int_B |\nabla f(x)|^2\ \mathrm{d}x\right)^{1/2}, \qquad f \in W_0^{1,2}(B), \tag{1.1}$$

where B is the unit ball in $\mathbb{R}^N$ and V is a weight in B, that is, a.e. non-negative and locally integrable function in B; this is a local version of

$$\left(\int_{\mathbb{R}^N} f(x)^2 V(x)\ \mathrm{d}x\right)^{1/2} \leq c\left(\int_{\mathbb{R}^N} |\nabla f(x)|^2\ \mathrm{d}x\right)^{1/2}, \qquad f \in W^{1,2}(\mathbb{R}^N). \tag{1.2}$$

All functions are supposed to be real-valued (complex-valued functions can be considered, too) and various constants independent of f will be denoted by the same generic symbol c, C etc. if no misunderstanding can arise. Variants and generalizations of the above inequalities have been intensively studied during last decades. They appear under various names as the *trace inequality* or the *uncertainty principle* and they have many relevant applications in analysis. It would be a difficult task to collect even the most important references and we shall make no attempt to do that. We shall just recall several basic facts and explain our motivation.

Necessary and sufficient conditions for the imbedding of $W^{1,p}$ into $L^q(V)$ have been studied in a number of papers, we cite at least Adams' inequality in [1], Maz'ya's pioneering works using capacities, [14], [15]. For $p = q = 2$ and $N \geq 3$, a necessary and sufficient condition is due to Kerman and Sawyer [10] – this is connected with Sawyer's necessary and sufficient conditions for validity of two weight inequalities for the Riesz potentials, see [17].

Due to the nature of these two-weight conditions (which require an information on the acting of Riesz potentials on weights in question) and of capacities, of importance are sufficient conditions (close to necessary ones as much as possible of course) in amenable

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terms of various classes and/or spaces of function. Fefferman in [6] gave the following sufficient condition: Let us recall that the *Fefferman-Phong class* F_p, $1 \le p \le N/2$, consists of functions V such that

$$\|V\|_{F_p} = \sup_{\substack{x\in\mathbb{R}^N\\ r>0}} r^2\biggl(\frac{1}{|B(x,r)|}\int_{B(x,r)} |V(y)|^p \,\mathrm{d}y\biggr)^{1/p} < \infty.$$

THEOREM 1.1 (Fefferman [6]). *Let* $N \ge 3$, $1 < p \le N/2$, *and* $V \in F_p$. *Then* (1.2) *holds.*

Note that Chiarenza and Frasca [4] gave a very fine alternative proof with help of properties of the maximal operator. We have $F_{p_2} \subset F_{p_1}$ for $1 \le p_1 \le p_2 \le N/2$, and $F_{N/2} = L^{N/2}$. If we restrict ourselves to balls $B(x,r)$, $0 < r < \varepsilon_0$, we get the *Morrey space* $\mathcal{L}^{p,N-2p}$; recall that for $0 < \lambda \le N$ and $1 \le p < \infty$, the Morrey space $\mathcal{L}^{p,\lambda}$ is the collection of all $V \in L^p_{\mathrm{loc}}$ such that

$$\|V\|_{\mathcal{L}^{p,\lambda}} = \sup_{\substack{x\in\mathbb{R}^N\\ 0<r\le r_0}} r^{-\lambda/p}\biggl(\int_{B(x,r)} |V(y)|^p \,\mathrm{d}y\biggr)^{1/p} < \infty.$$

Inserting a 'hat function',

$$u(x) = (r - |x|)\chi_{B(0,r)}, \qquad x \in \mathbb{R}^N,$$

into (1.2) we see that $V \in \mathcal{L}^{1,1}$ in order that (1.2) holds. Nevertheless, as is well known, this is not sufficient for validity of (1.2). Further investigation shows that the situation near $\mathcal{L}^{1,N-2}$ is of rather delicate nature. Various refined conditions have been considered in the literature, see e.g. Zamboni [19], Di Fazio [5], and Kurata [12] (Stummel-Kato classes and so on).

For $N = 2$ there is the sufficient condition $V \in L\log L$ for (1.1) due to Gossez and Loulit in [8] and a more general condition in terms of Lorentz-Zygmund spaces and based on fine critical imbedding theorem due to Brezis and Wainger [3], see Krbec and Schott [11]; this is, however, strictly limited to planar domains.

We shall use the standard notation $\|\,.\,\|_{k,p}$ for the norm in $W^{k,p}$; if $k = 0$, then $W^{k,p} = L^p$ with the norm denoted by $\|\,.\,\|_p$. If V is a weight in a domain $G \subset \mathbb{R}^N$ then the *weighted Lebesgue space* $L^p(V) = L^p(V,G)$ is defined as the space of all measurable f in G with the finite norm $\|f|L^p(V)\| = \bigl(\int_G |f(x)|^p V(x)\,\mathrm{d}x\bigr)^{1/p}$; the *weighted Sobolev space* $W^{k,p}(V) = W^{k,p}(V,G)$ will then be the space of all functions with weak derivatives up to the order k and the finite norm $\sum_{|\alpha|\le k}\|D^\alpha f|L^p(V)\|$. If f is a measurable function in $\mathbb{R}^N$, then f^* will denote its *non-increasing rearrangement*. The symbol $L^{p,q}$ will stand for the usual *Lorentz space* $(1 \le p, q \le \infty)$.

2. Weighted imbeddings. We shall discuss weighted consequences of the Sobolev imbedding theorem and of the Gross logarithmic inequality. A very suitable auxiliary tool for that will be the general imbedding theorem due to Ishii [9]. The *Young function* in the following is an even, convex function $\Phi : \mathbb{R} \to [0,\infty)$ such that $\Phi(0) = 0$ and $\lim_{t\to\infty}\Phi(t) = \infty$. If w is a weight on a measurable set $G \subset \mathbb{R}^N$, then we consider the *weighted modular* $\rho(f,\Phi,w) = \int_G \Phi(f(x))w(x)\,\mathrm{d}x$ and the corresponding Luxemburg or Orlicz norm,

$$\|f|L_\Phi(w)\| = \inf\{\mu :\ \rho(f/\mu,\Phi,w) \le 1\},$$

giving the *weighted Orlicz space* $L_\Phi(w)$. The symbol $L^p \log L$ will denote the Orlicz space with the generating Young function $t \mapsto |t|^p \log(e+t)$, $t \in \mathbb{R}$, and $L_{\exp t^\alpha}$ for $\alpha > 0$ will stand for the space with the Young function $t \mapsto \exp(|t|^\alpha) - 1$, $t \in \mathbb{R}$; for $\alpha = 1$ we shall simply write $L_{\exp}$.

We state the above mentioned imbedding theorem (see [9] and [16]) in a slightly modified form, suitable for our purposes. Note that the norm of the imbedding in the theorem is independent of the dimension since it is a reformulation of an abstract theorem, which holds true in general Musielak-Orlicz spaces.

Proposition 2.1 (Ishii). *Let u and v be weights in a measurable set $G \subset \mathbb{R}^N$, and let Φ and Ψ be Young functions. Then $L_\Phi(u) \hookrightarrow L_\Psi(v)$ if and only if there exists $K > 1$ such that the function*

$$x \mapsto \sup_{t>0} \left[\Psi(t)v(x) - \Phi(Kt)u(x)\right], \qquad x \in G,$$

is integrable over G.

Since we are interested in large N's we shall tacitly assume that $N \geq 3$ in the following to avoid unnecessary technicalities.

First of of let us briefly discuss a straightforward approach based on Sobolev imbeddings. Invoking Theorem 1.1 we see that $V \in L^{N/2}$ is a sufficient condition for (1.1); this is the "worst" case possible. One can do a little bit better: Since $W^{1,2}(B)$ is imbedded into the Lorentz space $L^{2N/(N-2),2}$ we have

$$\begin{aligned}\int_B f(x)^2 V(x)\,\mathrm{d}x &\leq \int_0^{|B|} f^*(t)^2 V^*(t)\,\mathrm{d}t\\ &\leq \int_0^{|B|} t^{(N-2)/N} f^*(t)^2 t^{2/N} V^*(t)\,\frac{\mathrm{d}t}{t}\\ &\leq \sup_{0<s<\infty} s^{2/N} V^*(s) \int_0^{|B|} \left(t^{(N-2)/2N} f^*(t)\right)^2\,\frac{\mathrm{d}t}{t},\end{aligned}$$

where we have used the Hardy-Littlewood rearrangement inequality Hence (1.1) holds if $V \in L^{N/2,\infty}$. In particular, $V \in L_{\exp t^2}$ is sufficient for (1.1) in any $\mathbb{R}^N$. Nevertheless, a dimension-free imbedding would require a detailed inspection of the behaviour of the imbedding constants and also of the equivalence of the exponential norm of V with the asymptotic estimates for the $L^{N/2,\infty}$ norms in dependence on N. We shall not pursue this line here.

Instead, we shall employ the well-known dimension-free estimate for functions in $W^{1,2}(\mathbb{R}^N)$ due to Gross, usually called the Gross logarithmic inequality (see e.g. [13] for a detailed discussion). It can be formulated as follows:

$$\int_{\mathbb{R}^N} f(x)^2 \log\left(\frac{f(x)^2}{\|f\|_2^2}\right)\,\mathrm{d}x + N\|f\|_2^2 \leq \frac{1}{\pi}\int_{\mathbb{R}^N} |\nabla f(x)|^2\,\mathrm{d}x. \tag{2.1}$$

If, say, $\|f|W^{1,2}(\mathbb{R}^N)\| = 1/2$, we obtain from (2.1) that

$$\int_{\mathbb{R}^N} f(x)^2 \log|f(x)|\,\mathrm{d}x \leq \frac{1}{2\pi}\int_{\mathbb{R}^N} |\nabla f(x)|^2\,\mathrm{d}x \tag{2.2}$$

(since under our assumption $\log\|f\|_2 \leq 0$).

Note also that in [2] Adams considered more general and dimension dependent inequalities (with norms taken with respect to the Gaussian measure $\exp(-|x|^2)\,\mathrm{d}x$).

Theorem 2.2. *Let $N \geq 3$, and $V \in L_{\exp}(B)$. Then there exists $c > 0$ independent of N such that*

$$\int_B f(x)^2 V(x) \, \mathrm{d}x \leq c \|f|W^{1,2}(B)\|^2$$

for all $f \in W_0^{1,2}(B)$.

Proof. Let $f \in W_0^{1,2}(B)$, $\|f|W^{1,2}(B)\| = 1/2$. Since $W_0^{1,2}(B)$ is a lattice we can suppose (considering $|f|$ instead of f) that $f \geq 0$ a.e. in B. Denote the extension of f by zero to the whole of $\mathbb{R}^N$ by the same symbol. Consider $\widetilde{f}(x) = f(x) + \varepsilon h(x)$, where

$$h(x) = \begin{cases} 1 & \text{if } |x| \leq 1, \\ |x|^{-\alpha} & \text{if } |x| > 1, \end{cases}$$

where $\alpha > (N-2)/2$. Our first goal will be to show that

$$\int_{\mathbb{R}^N} (f(x) + \varepsilon h(x))^2 \log(1 + f(x) + \varepsilon h(x)) \, \mathrm{d}x \leq c \|f|W^{1,2}(B)\|^2 \tag{2.3}$$

with some constant c independent of the dimension and (small) ε. Then Fatou's lemma applied to (2.3) for $\varepsilon \to 0_+$ will give

$$\int_B (f(x))^2 \log(1 + f(x)) \, \mathrm{d}x \leq c \|f|W^{1,2}(B)\|^2. \tag{2.4}$$

The next step will be then to derive the desired weighted inequality from (2.4).

Let us turn our attention to (2.3). We have

$$\begin{aligned} &\int_{\mathbb{R}^N} (f(x) + \varepsilon h(x))^2 \log(1 + f(x) + \varepsilon h(x)) \, \mathrm{d}x \\ &\quad = \int_{f(x) \geq 2} (f(x) + \varepsilon h(x))^2 \log(1 + f(x) + \varepsilon h(x)) \, \mathrm{d}x \\ &\qquad + \int_{0 < f(x) < 2} (f(x) + \varepsilon h(x))^2 \log(1 + f(x) + \varepsilon h(x)) \, \mathrm{d}x \\ &\qquad + \int_{f \equiv 0} (\varepsilon h(x))^2 \log(1 + \varepsilon h(x)) \, \mathrm{d}x \\ &\quad \leq \int_{f(x) \geq 2} (f(x) + \varepsilon)^2 \log(1 + f(x) + \varepsilon) \, \mathrm{d}x \\ &\qquad + \int_{0 < f(x) < 2} (f(x) + \varepsilon)^2 \log(1 + f(x) + \varepsilon) \, \mathrm{d}x \\ &\qquad + \int_B \varepsilon^2 \log(1 + \varepsilon) \, \mathrm{d}x + \int_{\mathbb{R}^N \setminus B} \left(\frac{\varepsilon}{|x|^\alpha}\right)^2 \log\left(1 + \frac{\varepsilon}{|x|^\alpha}\right) \mathrm{d}x \\ &\quad = I_1(\varepsilon) + I_2(\varepsilon) + I_3(\varepsilon) + I_4(\varepsilon). \end{aligned} \tag{2.5}$$

By virtue of (2.2), since $\log(1 + f(x) + \varepsilon) \leq 2 \log(f(x) + \varepsilon)$ if $f(x) \geq 2$,

$$I_1(\varepsilon) = \int_{f(x) \geq 2} (f(x) + \varepsilon)^2 \log(1 + f(x) + \varepsilon) \, \mathrm{d}x \leq \frac{1}{\pi} \int_{\mathbb{R}^N} |\nabla (f(x) + \varepsilon h(x))|^2 \, \mathrm{d}x.$$

For the right hand side there is the elementary estimate

$$\int_{\mathbb{R}^N} |\nabla(f(x)+\varepsilon h(x))|^2 \,\mathrm{d}x \le c\|f|W^{1,2}\|^2 + c\varepsilon^2 \int_{\mathbb{R}^N\setminus B} |\nabla h(x)|^2 \,\mathrm{d}x.$$

Since

$$\left|\frac{\partial}{\partial x_i}|x|^{-\alpha}\right| \le \alpha|x|^{-(\alpha+1)}, \qquad |x|\ge 1,$$

we have $|\nabla h(x)|^2 \le \alpha^2|x|^{-(2\alpha+2)}$ and for $\alpha > -1+N/2$ we get, passing to polar coordinates,

$$\int_{|x|\ge 1} |\nabla h(x)|^2 \,\mathrm{d}x \le \alpha^2 \int_{|x|\ge 1} |x|^{-(2\alpha+2)} \,\mathrm{d}x = \omega_N \alpha^2 \frac{1}{2\alpha+2-N},$$

where ω_N is the surface measure of the unit sphere, that is,

$$\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}.$$

By Stirling's formula,

$$\Gamma(N/2) \sim \mathrm{e}^{-N/2}(N/2)^{N/2-1/2}.$$

Put $\alpha = N$ for the rest of the proof. Then we obtain

$$\begin{aligned}
\int_{|x|\ge 1} |\nabla h(x)|^2 \,\mathrm{d}x &\le \frac{2\pi^{N/2}}{\Gamma(N/2)}\frac{N^2}{2N+2-N} \le c\frac{2\pi^{N/2}}{\Gamma(N/2)}N \\
&\le c\frac{2\pi^{N/2}\,\mathrm{e}^{N/2}\,N}{(N/2)^{N/2-1/2}} \sim \frac{2^{N/2}\pi^{N/2}\,\mathrm{e}^{N/2}\,N^{3/2}}{N^{N/2}} \\
&= \left(\frac{(2\pi\,\mathrm{e})^N N^3}{N^N}\right)^{1/2},
\end{aligned}$$

which is bounded uniformly with respect to N (and even tends to 0 when $N\to\infty$). Altogether

$$I_1(\varepsilon) \le c\|f|W^{1,2}\| + c\varepsilon^2. \tag{2.6}$$

We estimate the second integral. We shall make use of the well-known behaviour of the best constant for the Sobolev imbedding; there holds

$$\|f|L^{2N/(N-2)}(\mathbb{R}^N)\| \le A_N\|f|W^{1,2}(\mathbb{R}^N)\|, \tag{2.7}$$

where

$$A_N^{-1} = \sqrt{\pi}N^{1/2}(N-2)^{1/2}\left(\frac{\Gamma(N/2)}{\Gamma(N)}\right)^{1/N}$$

(see e.g. [18]). By Stirling's formula,

$$A_N^{-1} \sim N^{1/2}. \tag{2.8}$$

Hence by Hölder's inequality,

$$\begin{aligned} I_2(\varepsilon) &= \int_{0<f(x)<2} (f(x)+\varepsilon)^2 \log(1+f(x)+\varepsilon)\,dx \\ &\le c\Big(\int_{0<f(x)<2} (f(x)+\varepsilon)^{2N/(N-2)}\Big)^{(N-2)/N} |B|^{2/N} \\ &\le cN^{-1}|B|^{2/N} \le cN^{-1}\left(\frac{\pi^{N/2}}{\Gamma(1+N/2)}\right)^{2/N} \\ &\sim N^{-1}N^{-1} \sim N^{-2}, \end{aligned} \tag{2.9}$$

where the first equivalence on the last line follows again from Stirling's formula.

Finally, for small ε, the third and the fourth integrals can be estimated as follows:

$$\begin{aligned} I_3(\varepsilon)+I_4(\varepsilon) &\le \int_B \varepsilon^2 \log\frac{1}{\varepsilon}\ \mathrm{d}x + \int_{\mathbb{R}^N\setminus B} \left(\varepsilon|x|^{-\alpha}\right)^2 \log\frac{|x|^\alpha}{\varepsilon}\ \mathrm{d}x \\ &\le |B|\varepsilon^2\log\frac{1}{\varepsilon} + c\omega_N\varepsilon^2\int_1^\infty r^{N-1-2\alpha}\log\frac{r^\alpha}{\varepsilon}\,dr \end{aligned}$$

which is, after putting $\varepsilon=\delta^\alpha$, and with $\alpha=N$,

$$\begin{aligned} &= |B|\varepsilon^2\log\frac{1}{\varepsilon} + c\omega_N\delta^{2N}N\int_1^\infty r^{N-2N-1}\log\frac{r}{\delta}\,dr \\ &= |B|\varepsilon^2\log\frac{1}{\varepsilon} + c\omega_N\delta^{2N-1}N\int_1^\infty r^{-N}\left(\frac{r}{\delta}\right)^{-1}\log\frac{r}{\delta}\,dr \\ &\le c|B|\varepsilon^{2-1/N}\frac{N}{N-1}. \end{aligned}$$

Hence the left hand side of (2.3) is finite. Fatou's lemma gives then

$$\int_B f(x)^2\log(1+f(x))\ \mathrm{d}x \le c\|f|W_0^{1,2}(B)\|^2, \tag{2.10}$$

where c is independent of the dimension N.

Since the modular and the norm convergence in $L^2\log L(B)$ are equivalent we arrive at the imbedding $W_0^{1,2}(B)\hookrightarrow L^2\log L$ in any $\mathbb{R}^N$, $N\in\mathbb{N}$, with the norm independent of N.

Now our problem reduces to establishing a sufficient condition for the imbedding $L^2\log L(B)\hookrightarrow L^2(V,B)$, where $L^2\log L(B)$ is the Orlicz space generated by the Young function $t\mapsto t^2\log(1+|t|)$. Ishii's theorem gives a necessary and sufficient condition for that, namely, integrability of the function

$$\sup_{t>0}[t^2V(x)-Kt^2\log(1+Kt)], \qquad x\in B, \tag{2.11}$$

over B, for some $K>1$. Let us rewrite the function in (2.11) as

$$\sup_{t>0}[tV(x)-Kt\log(1+Kt^{1/2})]. \tag{2.12}$$

By virtue of the Young inequality the condition is $V\in L_{\widetilde{\Psi}}(B)$, where $\widetilde{\Psi}$ is the complementary function to $\Psi(t)=|t|\log(1+t^{1/2})$. Note that $\Psi(t)\sim|t|\log(1+t)$ and it is well known that the complementary function is equivalent to $t\mapsto\exp|t|-1$. □

REMARK 2.1. A closer inspection of the proof shows that THEOREM 2.2 remains to be true in $\mathbb{R}^N$. In particular, the chain of the inequalities in (2.9) does not hold then and one has to replace it by an appropriate estimate for the function $f(x) + \varepsilon h(x)$ in $B \cap \{0 < f(x) < 2\}$ and in $(\mathbb{R}^N \setminus B) \cap \{0 < f(x) < 2\}$. The asymptotic estimate of $I_2(\varepsilon)$ by cN^{-3} does not hold, nevertheless, it is not difficult to show that $I_2(\varepsilon)$ is bounded and consequently one can pass to the lim inf as before. The integrals $I_3(\varepsilon)$ and $I_4(\varepsilon)$ can be treated with similarly.

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