M. Geissert; M. Hieber

L^p-theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle

In: Marek Fila and Karol Mikula and Pavol Quittner (eds.): Proceedings of Equadiff 11, International Conference on Differential Equations. Czecho-Slovak series, Bratislava, July 25-29, 2005, [Part 1] Contributions of plenary speakers and minisymposia organizers. Comenius University Press, Bratislava, 2007. Presented in electronic form on the Internet. pp. 69--76.

Persistent URL: `http://dml.cz/dmlcz/700394`

Acta Math. Univ. Comenianae
Vol. LXXVI, 1(2007), pp. 69–76
Proceedings of Equadiff 11

L^p-THEORY OF THE NAVIER-STOKES FLOW IN THE EXTERIOR OF A MOVING OR ROTATING OBSTACLE

M. GEISSERT and M. HIEBER

Abstract. In this paper we describe two recent approaches for the L^p-theory of the Navier-Stokes flow in the exterior of a moving or rotating obstacle.

1. Introduction

Consider a compact set $O \subset \mathbb{R}^n$, the obstacle, with boundary $\Gamma := \partial O$ of class $C^{1,1}$. Set $\Omega := \mathbb{R}^n \backslash O$. For $t > 0$ and a real $n \times n$-matrix M we set

$$\Omega(t) := \{y(t) = e^{tM}x, x \in \Omega\} \text{ and } \Gamma(t) := \{y(t) = e^{tM}x, x \in \Gamma\}.$$

Then the motion past the moving obstacle O is governed by the equations of Navier-Stokes given by

$$\begin{array}{rcll} \partial_t w - \Delta w + w \cdot \nabla w + \nabla q &=& 0, & \text{in } \Omega(t) \times \mathbb{R}_+, \\ \nabla \cdot w &=& 0, & \text{in } \Omega(t) \times \mathbb{R}_+, \\ w(y,t) &=& My, & \text{on } \Gamma(t) \times \mathbb{R}_+, \\ w(y,0) &=& w_0(y), & \text{in } \Omega. \end{array} \tag{1}$$

Here $w = w(y,t)$ and $q(y,t)$ denote the velocity and the pressure of the fluid, respectively. The boundary condition on $\Gamma(t)$ is the usual no-slip boundary condition. Quite a few articles recently dealt with the equation above, see [**2**], [**3**], [**4**], [**5**], [**6**], [**8**], [**10**], [**11**], [**15**], [**16**].

In this paper, we describe two approaches to the above equations for the L^p-setting where $1 < p < \infty$. The basic idea for both approaches is to transfer the problem given on a domain $\Omega(t)$ depending on t to a fixed domain. The first transformation described in the following Section 2 yields additional terms in the equations which are of Ornstein-Uhlenbeck type. We shortly describe the techniques used in [**15**] and [**12**] in order to construct a local mild solution of (1).

In contrast to the first transformation, the second one, inspired by [**17**] and [**6**], allows to invoke maximal L^p-estimates for the classical Stokes operator in exterior domains and like this we obtain a unique strong solution to (1). This approach is described in section 3.

Received December 1, 2005.
2000 *Mathematics Subject Classification.* Primary 35Q30, 76D03.
Key words and phrases. Navier-Stokes, rotating obstacle, mild and strong solutions.
Supported by the DFG-Graduiertenkolleg 853.

2. Mild solutions

In this section we construct mild solutions to the Navier-Stokes problem (1). To do this we first transform the equations (1) to a fixed domain. Let Ω, $\Omega(t)$ and $\Gamma(t)$ be as in the introduction and suppose that M is unitary. Then by the change of variables $x = \mathrm{e}^{-tM}y$ and by setting $v(x,t) = e^{-tM}w(\mathrm{e}^{tM}x,t)$ and $p(x,t) = q(\mathrm{e}^{tM}x,t)$ we obtain the following set of equations defined on the fixed domain Ω:

$$
(2)\qquad \begin{aligned}
\partial_t v - \Delta v + v\cdot\nabla v - Mx\cdot\nabla v + Mv + \nabla p &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\
\nabla\cdot v &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\
v(x,t) &= Mx, && \text{on } \Gamma\times\mathbb{R}_+,\\
v(x,0) &= w_0(x), && \text{in } \Omega.
\end{aligned}
$$

Note that the coefficient of the convection term $Mx\cdot\nabla u$ is unbounded, which implies that this term cannot be treated as a perturbation of the Stokes operator.

This problem was first considered by Hishida in $L^2_\sigma(\Omega)$ for $\Omega\subset\mathbb{R}^3$ and $Mx = \omega\times x$ with $\omega = (0,0,1)^T$ in [**15**] and [**16**]. The L^p-theory was developed by Heck and the authors in [**12**] even for general M.

We will construct mild solutions for $w_0\in L^p_\sigma(\Omega)$, $p\geq n$, to the problem (2) with Kato's iteration (see [**18**]).

The starting point is the linear problem

$$
(3)\qquad \begin{aligned}
\partial_t u - \Delta u - Mx\cdot\nabla u + Mu + b\cdot\nabla u + u\cdot\nabla b + \nabla p &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\
\nabla\cdot u &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\
u &= 0, && \text{on } \Gamma\times\mathbb{R}_+,\\
u(x,0) &= w_0(x), && \text{in } \Omega,
\end{aligned}
$$

where $b\in C_c^\infty(\overline{\Omega})$. The additional term $b\cdot\nabla u + u\cdot\nabla b$ simplifies the treatment of the Navier-Stokes problem (see (11) below). We will first show that the solution of (3) is governed by a C_0-semigroup on $L^p_\sigma(\Omega)$. More precisely, let $L_{\Omega,b}$ be defined by

$$
\begin{aligned}
L_{\Omega,b}u &:= P_\Omega\mathcal{L}_b u\\
D(L_{\Omega,b}) &:= \{u\in W^{2,p}(\Omega)\cap W^{1,p}_0(\Omega)\cap L^p_\sigma(\Omega) : Mx\cdot\nabla u\in L^p(\Omega)\},
\end{aligned}
$$

where $\mathcal{L}_b u := \Delta u + Mx\cdot\nabla u - Mu + b\cdot\nabla u + u\cdot\nabla b$. Then the following theorem is proved in [**12**].

Theorem 2.1. *Let $1 < p < \infty$ and let $\Omega\subset\mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary. Assume that $\operatorname{tr} M = 0$ and $b\in C_c^\infty(\overline{\Omega})$. Then the operator $L_{\Omega,b}$ generates a C_0-semigroup $T_{\Omega,b}$ on $L^p_\sigma(\Omega)$.*

Sketch of the proof. The proof is devided into several steps. First it is shown that $L_{\Omega,b}$ is the generator of an C_0-semigroup $T_{\Omega,b}$ on $L^2_\sigma(\Omega)$. Then a-priori L^p-estimates for $T_{\Omega,b}$ are proved. Once we have shown this we can easily define a consistent family of semigroups $T_{\Omega,b}$ on $L^p_\sigma(\Omega)$ for $1 < p < \infty$. In the last step the generator of $T_{\Omega,b}$ on $L^p_\sigma(\Omega)$ is identified to be $L_{\Omega,b}$.

We start by showing that $L_{\Omega,b}$ is the generator of a C_0-semigroup on $L^2_\sigma(\Omega)$. Choose $R > 0$ such that $\operatorname{supp} b \cup \Omega^c \subset B_R(0) = \{x \in \mathbb{R}^n : |x| < R\}$. We then set

$$\begin{aligned} D &= \Omega \cap B_{R+5}(0), \\ K_1 &= \{x \in \Omega : R < |x| < R+3\}, \\ K_2 &= \{x \in \Omega : R+2 < |x| < R+5\}. \end{aligned}$$

Denote by B_i for $i \in \{1,2\}$ Bogovskiĭ's operator (see [**1**], [**9**, Chapter III.3], [**13**]) associated to the domain K_i and choose cut-off functions $\varphi, \eta \in C^\infty(\mathbb{R}^n)$ such that $0 \le \varphi, \eta \le 1$ and

$$\varphi(x) = \begin{cases} 0, & |x| \le R+1, \\ 1, & |x| \ge R+2, \end{cases} \quad \text{and} \quad \eta(x) = \begin{cases} 1, & |x| \le R+3, \\ 0, & |x| \ge R+4. \end{cases}$$

For $f \in L^p_\sigma(\Omega)$ we denote by f^R the extension of f by 0 to all of $\mathbb{R}^n$. Then, since $C^\infty_{c,\sigma}(\Omega)$ is dense in $L^p_\sigma(\Omega)$, $f^R \in L^p_\sigma(\mathbb{R}^n)$. Furthermore, we set $f^D = \eta f - B_2((\nabla\eta)f)$. Since $\int_{K_2}(\nabla\eta)f = 0$ it follows from [**9**, Chapter III.3] that $f^D \in L^p_\sigma(D)$.

By the perturbation theorem for analytic semigroups there exists $\omega_1 \ge 0$ such that for $\lambda > \omega_1$ there exist functions u^D_λ and p^D_λ satisfying the equations

$$\begin{aligned} (\lambda - \mathcal{L}_b)u^D_\lambda + \nabla p^D_\lambda &= f^D, && \text{in } D \times \mathbb{R}_+, \\ \nabla \cdot u^D_\lambda &= 0, && \text{in } D \times \mathbb{R}_+, \\ u^D_\lambda &= 0, && \text{on } \partial D \times \mathbb{R}_+. \end{aligned} \tag{4}$$

Moreover, by [**14**, Lemma 3.3 and Prop. 3.4], there exists $\omega_2 \ge 0$ such that for $\lambda > \omega_2$ there exists a function u^R_λ satisfying

$$\begin{aligned} (\lambda - \mathcal{L}_0)u^R_\lambda &= f^R, && \text{in } \mathbb{R}^n \times \mathbb{R}_+, \\ \nabla \cdot u^R_\lambda &= 0, && \text{in } \mathbb{R}^n \times \mathbb{R}_+. \end{aligned} \tag{5}$$

For $\lambda > \max\{\omega_1,\ \omega_2\}$ we now define the operator $U_\lambda : L^p_\sigma(\Omega) \to L^p_\sigma(\Omega)$ by

$$U_\lambda f = \varphi u^R_\lambda + (1-\varphi)u^D_\lambda + B_1(\nabla\varphi(u^R_\lambda - u^D_\lambda)), \tag{6}$$

where u^R_λ and u^D_λ are the functions given above, depending of course on f. By definition, we have

$$U_\lambda f \in \{v \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega) \cap L^p_\sigma(\Omega) : Mx \cdot \nabla v \in L^p_\sigma(\Omega)\}. \tag{7}$$

Setting $P_\lambda f = (1-\varphi)p^D_\lambda$, we verify that $(U_\lambda f,\ P_\lambda f)$ satisfies

$$\begin{aligned} (\lambda - \mathcal{L}_b)U_\lambda f + \nabla P_\lambda f &= f + T_\lambda f, && \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot U_\lambda f &= 0, && \text{in } \Omega \times \mathbb{R}_+, \\ U_\lambda f &= 0, && \text{on } \partial\Omega \times \mathbb{R}_+, \end{aligned}$$

where T_λ is given by

$$\begin{aligned} T_\lambda f = & -2(\nabla\varphi)\nabla(u^R_\lambda - u^D_\lambda) - (\Delta\varphi + Mx\cdot(\nabla\varphi))(u^R_\lambda - u^D_\lambda) + (\nabla\varphi)p^D_\lambda \\ & + (\lambda - \Delta - Mx\cdot\nabla + M)B_1((\nabla\varphi)(u^R_\lambda - u^D_\lambda)). \end{aligned}$$

It follows from [**12**, Lemma 4.4] that for $\alpha \in (0, \frac{1}{2p'})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, there exists a strongly continuous function $H : (0,\infty) \to \mathcal{L}(L^p_\sigma(\Omega))$ satisfying

$$\|H(t)\|_{\mathcal{L}(L^p_\sigma(\Omega))} \leq Ct^{\alpha-1}\mathrm{e}^{\tilde{\omega}t}, \quad t > 0 \tag{8}$$

for some $\tilde{\omega} \geq 0$ and $C > 0$ such that $\lambda \mapsto P_\Omega T_\lambda$ is the Laplace Transform of H. We thus easily calculate

$$\|P_\Omega T_\lambda\|_{\mathcal{L}(L^p_\sigma(\Omega))} \leq C\lambda^{-\alpha}, \quad \lambda > \omega.$$

Therefore, $R_\lambda := U_\lambda \sum_{j=0}^\infty (P_\Omega T_\lambda)^j$ exists for λ large enough and $(\lambda - L_b)R_\lambda f = f$ for $f \in L^2_\sigma(\Omega)$. Since $L_{\Omega,b}$ is dissipative in $L^2_\sigma(\Omega)$, $L_{\Omega,b}$ generates a C_0-semigroup $T_{\Omega,b}$ on $L^2_\sigma(\Omega)$. Moreover, we have the representation

$$T_{\Omega,b}(t)f = \sum_{n=0}^\infty T_n(t)f, \quad f \in L^2_\sigma(\Omega), \tag{9}$$

where $T_n(t) := \int_0^t T_{n-1}(t-s)H(s)\,\mathrm{d}s$ for $n \in \mathbb{N}$ and

$$T_0(t) = \varphi T_R(t)f^R + (1-\varphi)T_{D,b}(t)f^D + B_1((\nabla\varphi)(T_R(t)f^R - T_{D,b}(t)f^D)), \quad t \geq 0.$$

Here T_R denotes the semigroup on $L^p_\sigma(\mathbb{R}^n)$ generated by $L_{\mathbb{R}^n,0}$ and $T_{D,b}$ denotes the semigroup on $L^p_\sigma(D)$ generated by $L_{D,b}$. Note that $\lambda \mapsto U_\lambda$ is the Laplace Transform of T_0. Since the right hand side of the representation (9) is well defined and exponentially bounded in $L^p_\sigma(\Omega)$ by [**12**, Lemma 4.6], we can define a family of consistent semigroups $T_{\Omega,b}$ on $L^p(\Omega)$ for $1 < p < \infty$. Finally, the generator of $T_{\Omega,b}$ on $L^p(\Omega)$ is $L_{\Omega,b}$ which can be proved by using duality arguments (cf. [**12**, Theorem 4.1]). □

Remark 2.2. (a) The semigroup $T_{\Omega,b}$ is not expected to be analytic since, by [**16**, Proposition 3.7], the semigroup $T_{\mathbb{R}^3}$ in $\mathbb{R}^3$ is not analytic.

(b) As the cut-off function φ is used for the localization argument similarly to [**15**] the purpose of η is to ensure that $f_D \in L^p_\sigma(\Omega)$. This is essential to establish a decay property in λ for the pressure P^D_λ (cf. [**12**, Lemma 3.5]) and T_λ.

(c) The crucial point for a-priori L^p-estimates for $T_{\Omega,b}$ on $L^2_\sigma(\Omega)$ is the existence of H satisfying (8).

Since L^p-L^q smoothing estimates for T_R and $T_{D,b}$ follow from [**14**, Lemma 3.3 and Prop. 3.4] and [**12**, Prop. 3.2], the representation of the semigroup $T_{\Omega,b}$ given by (9) and estimates for sums of convolutions of this type (cf. [**12**, Lemma 4.6]) yield the following proposition.

Proposition 2.3. *Let $1 < p < q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary. Assume that $\operatorname{tr} M = 0$ and $b \in C_c^\infty(\overline{\Omega})$. Then there exist constants $C > 0, \omega \geq 0$ such that for $f \in L^p_\sigma(\Omega)$*

(a) $\|T_{\Omega,b}(t)f\|_{L^q_\sigma(\Omega)} \leq Ct^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}\mathrm{e}^{\omega t}\|f\|_{L^p_\sigma(\Omega)}, \quad t > 0,$

(b) $\|\nabla T_{\Omega,b}(t)f\|_{L^p(\Omega)} \leq Ct^{-\frac{1}{2}}\mathrm{e}^{\omega t}\|f\|_{L^p_\sigma(\Omega)}, \quad t > 0.$

Moreover, for $f \in L^p_\sigma(\Omega)$

$$\|t^{\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)}T_{\Omega,b}(t)f\|_{L^q_\sigma(\Omega)} + \|t^{\frac{1}{2}}\nabla T_{\Omega,b}(t)f\|_{L^p(\Omega)} \to 0, \quad \textit{for} \quad t \to 0.$$

In order to construct a mild solution to (2) choose $\zeta \in C_c^\infty(\mathbb{R}^n)$ with $0 \le \zeta \le 1$ and $\zeta = 1$ near Γ. Further let $K \subset \mathbb{R}^n$ be a domain such that $\operatorname{supp} \nabla\zeta \subset K$. We then define $b : \mathbb{R}^n \to \mathbb{R}^n$ by

$$b(x) := \zeta M x - B_K((\nabla\zeta) M x), \tag{10}$$

where B_K is Bogovskiĭ's operator associated to the domain K. Then $\operatorname{div} b = 0$ and $b(x) = Mx$ on Γ. Setting $u := v - b$, it follows that u satisfies

$$\begin{array}{rcll} \partial_t u - \mathcal{L}_b u + \nabla p &=& F & \text{in } \Omega \times (0,T), \\ \nabla \cdot u &=& 0 & \text{in } \Omega \times (0,T), \\ u &=& 0 & \text{on } \Gamma \times (0,T), \\ u(x,0) &=& u_0(x) - b(x), & \text{in } \Omega, \end{array} \tag{11}$$

with $\nabla \cdot (u_0 - b) = 0$ in Ω and $F = -\Delta b - Mx \cdot \nabla b + Mb + b \cdot \nabla b$, provided u satisfies (2). Applying the Helmholtz projection P_Ω to (11), we may rewrite (11) as an evolution equation in $L^p_\sigma(\Omega)$:

$$\begin{array}{rcll} u' - L_{\Omega,b} u + P_\Omega(u \cdot \nabla u) &=& P_\Omega F, & 0 < t < T, \\ u(0) &=& u_0 - b. & \end{array} \tag{12}$$

Note that we need the compatibility condition $u_0(x) \cdot n = Mx \cdot n$ on $\partial\Omega$ to obtain $u_0 - b \in L^p_\sigma(\Omega)$. In the following, given $0 < T < \infty$, we call a function $u \in C([0,T); L^p_\sigma(\Omega))$ a *mild solution* of (12) if u satisfies the integral equation for $0 < t < T$

$$u(t) = T_{\Omega,b}(t)(u_0 - b) - \int_0^t T_{\Omega,b}(t-s) P_\Omega(u \cdot \nabla u)(s)\,\mathrm{d}s + \int_0^t T_{\Omega,b}(t-s) P_\Omega F(s)\,\mathrm{d}s.$$

Then the main result of [**12**] is the following theorem.

Theorem 2.4. *Let $n \ge 2$, $n \le p \le q < \infty$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary. Assume that $\operatorname{tr} M = 0$ and $b \in C_c^\infty(\overline{\Omega})$ and $u_0 - b \in L^p_\sigma(\Omega)$. Then there exist $T_0 > 0$ and a unique mild solution u of (12) such that*

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right)} u(t) \in C\left([0,T_0]\,; L^q_\sigma(\Omega)\right),$$

$$t \mapsto t^{\frac{n}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{2}} \nabla u(t) \in C\left([0,T_0]\,; L^q(\Omega)\right).$$

3. Strong solutions

In this section we construct strong solutions to problem (1) for $\Omega \subset \mathbb{R}^n$, $n \ge 2$ and $\operatorname{tr} M = 0$. The main difference to the method presented in the previous section is another change of variables. Indeed, we construct a change of variables which coincides with a simple rotation in a neighborhood of the rotating body but it equals to the identity operator far away from the rotating body. More precisely, let $X(\cdot,t):\mathbb{R}^n\to\mathbb{R}^n$ denote the time dependent vector field satisfying

$$\begin{aligned}\frac{\partial X}{\partial t}(y,t) &= -b(X(y,t)), \quad && y\in\mathbb{R}^n,\ t>0,\\ X(y,0) &= y, && y\in\mathbb{R}^n,\end{aligned}$$

where b is as in (10). Similarly to [**6**, Lemma 3.2], the vector field $X(\cdot,t)$ is a C^∞-diffeomorphism form Ω onto $\Omega(t)$ and $X\in C^\infty([0,\infty)\times\mathbb{R}^n)$. Let us denote the inverse of $X(\cdot,t)$ by $Y(\cdot,t)$. Then, $Y\in C^\infty([0,\infty)\times\mathbb{R}^n)$. Moreover, it can be shown that for any $T>0$ and $|\alpha|+k>0$ there exists $C_{k,\alpha,T}>0$ such that

$$\sup_{y\in\mathbb{R}^n,0\le t\le T}\left|\frac{\partial^k}{\partial t^k}\frac{\partial^\alpha}{\partial y^\alpha}X(y,t)\right|+\sup_{x\in\mathbb{R}^n,0\le t\le T}\left|\frac{\partial^k}{\partial t^k}\frac{\partial^\alpha}{\partial x^\alpha}Y(x,t)\right|\le C_{k,\alpha,T_0}. \tag{13}$$

Setting

$$v(x,t)=J_X(Y(x,t),t)w(Y(x,t),t), \quad x\in\Omega,\ t\ge 0,$$

where J_X denotes the Jacobian of $X(\cdot,t)$ and

$$p(x,t)=q(Y(x,t),t), \quad x\in\Omega,\ t\ge 0,$$

similarly to [**6**, Prop. 3.5] and [**17**], we obtain the following set of equations which are equivalent to (1).

$$\begin{aligned}\partial_t v-\mathcal{L}v+\mathcal{M}v+\mathcal{N}v+\mathcal{G}p &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\ \nabla\cdot v &= 0, && \text{in } \Omega\times\mathbb{R}_+,\\ v(x,t) &= Mx, && \text{on } \Gamma\times\mathbb{R}_+,\\ v(x,0) &= w_0(x), && \text{in } \Omega.\end{aligned} \tag{14}$$

Here

$$\begin{aligned}(\mathcal{L}v)_i &= \sum_{j,k=1}^n\frac{\partial}{\partial x_j}\left(g^{jk}\frac{\partial v_i}{\partial x_k}\right)+2\sum_{j,k,l=1}^n g^{kl}\Gamma^i_{jk}\frac{\partial v_j}{\partial x_l}\\ &\quad+\sum_{j,k,l=1}^n\left(\frac{\partial}{\partial x_k}(g^{kl}\Gamma^i_{jl})+\sum_{m=1}^n g^{kl}\Gamma^m_{jl}\Gamma^i_{km}\right)v_j,\\ (\mathcal{N}v)_i &= \sum_{j=1}^n v_j\frac{\partial v_i}{\partial x_j}+\sum_{j,k=1}^n\Gamma^i_{jk}v_jv_k,\\ (\mathcal{M}v)_i &= \sum_{j=1}^n\frac{\partial X_j}{\partial t}\frac{\partial v_i}{\partial x_j}+\sum_{j,k=1}^n\left(\Gamma^i_{jk}\frac{\partial X_k}{\partial t}+\frac{\partial X_i}{\partial x_k}\frac{\partial^2 Y_k}{\partial x_j\partial t}\right)v_j,\\ (\mathcal{G}p)_i &= \sum_{j=1}^n g^{ij}\frac{\partial p}{\partial x_j}\end{aligned}$$

with

$$\begin{aligned}g^{ij} &= \sum_{k=1}^n\frac{\partial X_i}{\partial y_k}\frac{\partial X_j}{\partial y_k}, \quad g_{ij}=\sum_{k=1}^n\frac{\partial Y_k}{\partial x_i}\frac{\partial Y_k}{\partial x_j}\ \text{and}\\ \Gamma^k_{ij} &= \frac{1}{2}\sum_{l=1}^n g^{kl}\left(\frac{\partial g_{il}}{\partial x_j}+\frac{\partial g_{jl}}{\partial x_i}+\frac{\partial g_{ij}}{\partial x_l}\right).\end{aligned}$$

The obvious advantage of this approach is that we do not have to deal with an unbounded drift term since all coefficients appearing in $\mathcal{L}$, $\mathcal{N}$, $\mathcal{M}$ and $\mathcal{G}$ are smooth and bounded on finite time intervals by (13). However, we have to consider a non-autonomous problem. Setting $u = v - b$, we obtain the following problem with homogeneous boundary conditions which is equivalent to (14).

$$\begin{aligned} \partial_t u - \mathcal{L}u + \mathcal{M}u + \mathcal{N}u + \mathcal{B}u + \mathcal{G}p &= F_b, && \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot u &= 0 && \text{in } \Omega \times \mathbb{R}_+, \\ u &= 0, && \text{on } \Gamma \times \mathbb{R}_+, \\ u(x,0) &= w_0(x) - b(x), && \text{in } \Omega. \end{aligned} \tag{15}$$

Here,

$$(\mathcal{B}u)_i = \sum_{j=1}^{n} \left(u_j \frac{\partial b_i}{\partial x_j} + b_j \frac{\partial u_i}{\partial x_j} \right) + 2 \sum_{j,k=1}^{n} \Gamma^i_{jk} u_j b_k, \quad F_b = \mathcal{L}b - \mathcal{M}b - \mathcal{N}b.$$

Since g^{ij} is smooth and $g^{ij}(\cdot, 0) = \delta_{ij}$ by definition, it follows from (13) that

$$\|g^{ij}(\cdot, t) - \delta_{ij}\|_{L^\infty(\Omega)} \to 0, \quad t \to 0. \tag{16}$$

In other words, $\mathcal{L}$ is a small perturbation of Δ and G is a small perturbation of ∇ for small times t. This motivates to write (15) in the following form.

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= F(u,p), && \text{in } \Omega \times \mathbb{R}_+, \\ \nabla \cdot u &= 0, && \text{in } \Omega \times \mathbb{R}_+, \\ u &= 0, && \text{on } \Gamma \times \mathbb{R}_+, \\ u(x,0) &= w_0(x) - b(x), && \text{in } \Omega, \end{aligned} \tag{17}$$

where $F(u,p) := (\mathcal{L} - \Delta)u - \mathcal{M}u - \mathcal{N}u + (\nabla - \mathcal{G})p - Bu + F_b$. We will use maximal L^p-regularity of the Stokes operator and a fixed point theorem to show the existence of a unique strong solution (u,p) of (15). More precisely, let

$$X_T^{p,q} := W^{1,p}(0,T; L^q(\Omega)) \cap L^p(0,T; D(A_q)) \times L^p(0,T; \widehat{W}^{1,p}(\Omega)),$$

where $D(A_q) := W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ is the domain of the Stokes operator. Then, by maximal L^p-regularity of the Stokes operator, Hölder's inequality and Sobolev's embedding theorems $\Phi : X_T^{p,q} \to X_T^{p,q}$, $\Phi((\tilde{u}, \tilde{p})) := (u,p)$ where (u,p) is the unique solution of

$$\begin{aligned} \partial_t u - \Delta u + \nabla p &= F(\tilde{u}, \tilde{p}), && \text{in } \Omega \times (0,T) \\ \nabla \cdot u &= 0, && \text{in } \Omega \times (0,T), \\ u &= 0, && \text{on } \Gamma \times (0,T), \\ u(x,0) &= w_0(x) - b(x), && \text{in } \Omega, \end{aligned}$$

is well-defined for $1 < p, q < \infty$ with $\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}$ and $T > 0$. Here, the restriction on p and q comes from the nonlinear term $\mathcal{N}$.

Finally, let $X_{T,\delta}^{p,q} := \{(u,p) \in X_T^{p,q} : \|(u,p) - (\hat{u}, \hat{p})\|_{X_T^{p,q}} \le \delta, u(0) = w_0 - b\}$ with $(\hat{u}, \hat{p}) = \Phi(\Phi(0,0))$. Then by (16), Hölder's inequality and Sobolev's embedding theorems, it can be shown that for small enough $\delta > 0$ and $T > 0$, $\Psi|_{X_{T,\delta}^{p,q}}$ is a contraction.

We summarize our considerations in the next theorem which is proved in [**7**]. Note that the cases $n = 2, 3$ and $p = q = 2$ were already proved in [**6**].

Theorem 3.1. *Let $1 < p, q < \infty$ such that $\frac{n}{2q} + \frac{1}{p} < \frac{3}{2}$ and let $\Omega \subset \mathbb{R}^n$ be an exterior domain with $C^{1,1}$-boundary. Assume that $\operatorname{tr} M = 0$ and that $w_0 - b \in (L^q_\sigma(\Omega), D(A_q))_{1-\frac{1}{p},p}$. Then there exist $T > 0$ and a unique solution $(u,p) \in X^{p,q}_T$ of problem* (15).

References

1. Bogovskiĭ M. E., *Solution of the first boundary value problem for an equation of continuity of an incompressible medium*. Dokl. Akad. Nauk SSSR **248** (1979), 1037–1040.

2. Banin A. Mahalov A. and Nicolaenko B., *Global regularity of 3D rotating Navier-Stokes equations for resonant domains*. Indiana Univ. Math. J. **48** (1999), 1133–1176.

3. ______, *3D Navier-Stokes and Euler equations with initial data characterized by uniformly large vorticity*. Indiana Univ. Math. J. **50** (2001), 1–35.

4. Borchers W., *Zur Stabilität und Faktorisierungsmethode für die Navier-Stokes-Gleichungen inkompressibler viskoser Flüssigkeiten*, Habilitationschrift Universität Paderborn, 1992.

5. Chen Z. and Miyakawa T., *Decay properties of weak solutions to a perturbed Navier-Stokes system in* $\mathbb{R}^n$. Adv. Math. Sci. Appl. **7** (1997), 741–770.

6. Cumsille P. and Tucsnak M., *Strong solutions for the Navier-Stokes flow in the exterior of a rotation obstacle*, Preprint, l'Institut Élie Cartan, 2004.

7. Dintelmann E., Geissert M. and Hieber M., *Strong solutions to the Navier-Stokes equations in the exterior of a moving or rotating obstacle*, in preparation.

8. Farwig R., Hishida T. and Müller D., *L^q-theory of a singular 'winding' integral operator arising from fluid dynamics*, Pacific J. Math. **215(2)** (2004), 297–312. TU Darmstadt, 2003.

9. Galdi G. P., *An Introduction to the Mathematical Theory of the Navier-Stokes equations. Vol. I*, Springer Tracts in Natural Philosophy, Vol. 38, Springer 1998.

10. ______, *Steady flow of a Navier-Stokes fluid around a rotating obstacle*, J. Elasticity **71 (1–3)** (2003), 1–31.

11. Galdi G. P., and Silvestre A. L., *Strong Solutions to the Navier-Stokes Equations Around a Rotating Obstacle*, Arch. Ration. Mech. Anal. **1763(3)** (2005), 331–350.

12. Geissert M., Heck H. and Hieber M., *L^p-theory of the Navier-Strokes flow in the exterior of a moving or rotating obstacle*, J. Reine Angew. Math., to appear.

13. ______, *On the equation* $\operatorname{div} u = f$ *and the Bogovskiĭ Operator*, in: G. Sweers (ed.), Functional Analysis and PDE, Birkhäuser, to appear.

14. Hieber M. and Sawada O., *The Navier-Stokes equations in* $\mathbb{R}^n$ *with linearly growing initial data*. Arch. Rational Mech. Anal., **175(2)** (2005), 269–285.

15. Hishida T., *An existence theorem for the Navier-Stokes flow in the exterior of a rotating obstacle*. Arch. Rat. Mech. Anal., **150** (1999), 307–348.

16. ______, *The Stokes operator with rotation effect in exterior domains*. Analysis, **19** (1999), 51–67.

17. Inoue A. and Wakimoto M., *On existence of solutions of the Navier-Stokes equation in a time dependent domain*. J. Fac. Sci. Univ. Tokyo Sect. IA Math., **24(2)** (1977), 303–319.

18. Kato T., *Strong L^p-solutions of Navier-Stokes equations in* $\mathbb{R}^n$ *with applications to weak solutions*. Math. Z., **187** (1984), 471–480.

M. Geissert, Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany,
e-mail: `geissert@mathematik.tu-darmstadt.de`

M. Hieber, Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, D-64289 Darmstadt, Germany,
e-mail: `hieber@mathematik.tu-darmstadt.de`