Acta Math. Univ. Comenianae
Vol. LXXVI, 1(2007), pp. 11–13
Proceedings of Equadiff 11

A REMARK ON THE LARGE TIME BEHAVIOR OF SOLUTIONS OF VISCOUS HAMILTON-JACOBI EQUATIONS

PH. SOUPLET

1. Introduction and main result

Consider the viscous Hamilton-Jacobi equation

$$\begin{cases} u_t - \Delta u = |\nabla u|^q, & t > 0, \quad x \in \mathbb{R}^N \\ u(0,x) = u_0(x), & \qquad\quad x \in \mathbb{R}^N, \end{cases} \tag{1}$$

where $q > 0$ and $u_0 \in C_b(\mathbb{R}^N)$. It is known [**6**] that (1) admits a unique classical solution, global for $t > 0$.

The large time behavior of solutions of problem (1) has been studied recently by several authors, see [**1**]–[**5**], [**7, 8**] and the references therein. In particular it was shown by Gilding [**5**] that the large time limits

$$\underline{\omega} := \liminf_{t\to\infty} v(x,t) \le \overline{\omega} := \limsup_{t\to\infty} v(x,t)$$

are independent of $x \in \mathbb{R}^N$. One of the main results of [**5**] is the following.

Theorem A. *Assume* $0 < q < 2$ *and* $u_0 \in C_b(\mathbb{R}^N)$. *Then* $\underline{\omega} = \overline{\omega}$.

It was known that Theorem A fails for the linear heat equation and, moreover, Gilding observed that it fails for $q = 2$. The aim of this short note is to show that the assumption $q < 2$ in Theorem A is actually necessary.

Theorem 1. *Assume* $q \ge 2$. *Then there exists* $u_0 \in C_b(\mathbb{R}^N)$ *such that* $\underline{\omega} < \overline{\omega}$.

Proof. It is known (see e. g. [**5**, Proposition H1]) that there exists $v_0 \in C^1(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N)$ such that the solution v of the heat equation

$$\begin{cases} v_t - \Delta v = 0, & t > 0, \quad x \in \mathbb{R}^N \\ v(0,x) = v_0(x), & \qquad\quad x \in \mathbb{R}^N \end{cases} \tag{2}$$

satisfies

$$\underline{\omega}^* := \liminf_{t\to\infty} v(x,t) < \overline{\omega}^* := \limsup_{t\to\infty} v(x,t), \quad x \in \mathbb{R}^N. \tag{3}$$

Received September 16, 2005.
2000 *Mathematics Subject Classification.* Primary 35K15, 35K55, 35B40.

Moreover, upon replacing v_0 by $\lambda v_0+\mu$ for suitable constants λ,μ, one can assume that

$$\underline{\omega}^*=0 \tag{4}$$

and

$$\|v_0\|_\infty\le 1/2,\qquad \|\nabla v_0\|_\infty\le 1/2.$$

Now, set

$$u_0(x):=\mathrm{e}^{v_0(x)}-1. \tag{5}$$

The function $w:=\mathrm{e}^v-1$ satisfies

$$\begin{cases} w_t-\Delta w=|\nabla w|^2, & t>0,\quad x\in\mathbb{R}^N\\ w(0,x)=u_0(x), & \qquad\quad x\in\mathbb{R}^N. \end{cases} \tag{6}$$

Let u be the solution of (1) with initial data u_0 defined by (5). We note that

$$\|\nabla u_0\|_\infty\le\|\nabla v_0\|_\infty\,\|\,\mathrm{e}^{v_0}\,\|_\infty\le(1/2)\,\mathrm{e}^{1/2}<1.$$

Since it is known (see e.g. [**5**, Lemma 2]) that $|\nabla u|$ satisfies a maximum principle, it follows that

$$|\nabla u|\le\|\nabla u_0\|_\infty<1\quad\text{ in } Q:=(0,\infty)\times\mathbb{R}^N.$$

Due to $q\ge 2$, we deduce that

$$u_t-\Delta u=|\nabla u|^q\le|\nabla u|^2\quad\text{ in } Q.$$

In view of (6), it follows from the comparison principle that

$$u\le w=\mathrm{e}^v-1\quad\text{ in } Q.$$

In particular, there holds

$$\underline{\omega}\le\mathrm{e}^{\underline{\omega}^*}-1=0. \tag{7}$$

But on the other hand, we have $u_0\ge v_0$ due to (5). In view of (2), the maximum principle implies that $u\ge v$, hence

$$\overline{\omega}\ge\overline{\omega}^*. \tag{8}$$

Combining (3), (4), (7) and (8), we conclude that

$$\overline{\omega}\ge\overline{\omega}^*>\underline{\omega}^*=0\ge\underline{\omega}$$

and the proof of Theorem 1 is complete. □

References

1. Ben-Artzi M., *Global properties of some nonlinear parabolic equations*, Nonlinear partial differential equations and their applications. Collège de France Seminar, **XIV** (Paris, 1997/1998), 57–67, Stud. Math. Appl. 31, North-Holland, Amsterdam, 2002.

2. Ben-Artzi M., Goodman J. and Levy A., *Remarks on a nonlinear parabolic equation*, Trans. Amer. Math. Soc. **352** (2000), 731–751.

3. Ben-Artzi M., Souplet Ph. and Weissler F. B., *The local theory for viscous Hamilton-Jacobi equations in Lebesgue spaces* J. Math. Pures Appl. **81** (2002), 343–378.

4. Benachour S., Karch G. and Laurençot Ph., *Asymptotic profiles of solutions to viscous Hamilton-Jacobi equations*, J. Math. Pures Appl., **83** (2004), 1275–1308.

5. Gilding B., *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$: Large-time behaviour*, J. Math. Pures Appl. **84** (2005), 753–785.

6. Gilding B., Guedda M. and Kersner R., *The Cauchy problem for $u_t = \Delta u + |\nabla u|^q$*, J. Math. Anal. Appl. **284** (2003), 733–755.

7. Laurençot Ph. and Souplet Ph., *On the growth of mass for a viscous Hamilton-Jacobi equation*, J. Anal. Math. **89** (2003), 367–383.

8. ______, *Optimal growth rates for a viscous Hamilton-Jacobi equation*, J. Evolution Equations **5** (2005), 123–135.

Ph. Souplet, Laboratoire Analyse Géométrie et Applications, Institut Galilée, Université Paris-Nord, 93430 Villetaneuse, France, *e-mail*: `souplet@math.univ-paris13.fr`