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Inertial Manifolds for Nonautonomous Dynamical Systems and for Nonautonomous Evolution Equations

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Abstract. In this paper we extend an abstract approach to inertial manifolds for nonautonomous dynamical systems. Our result on the existence of inertial manifolds requires only two geometrical assumptions, called cone invariance and squeezing property, and some additional technical assumptions like boundedness or smoothing properties.

In the second part of the paper we consider special nonautonomous dynamical systems, namely two-parameter semi-flows. As an application of our abstract approach and for reason of comparison with known results we verify the assumptions for semilinear nonautonomous evolution equations whose linear part satisfies an exponential dichotomy condition and whose nonlinear part is globally bounded and globally Lipschitz. Moreover, we apply our result on parabolic evolution equation with constant selfadjoint part. So we show that our abstract approach allows to obtain the sharp conditions in the autonomous case but they are applicable for the nonautonomous case, too.

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This is the final form of the paper.

1 Introduction

Let us consider a dissipative nonlinear evolution equation of the form

$$\dot{u} + Au = f(u)$$

in a Banach space $\mathcal{X}$, where A is a linear sectorial operator with compact resolvent and f is a nonlinear function. Such an evolution equation may be an ordinary differential equation ($\mathcal{X} = \mathbb{R}^n$) or the abstract formulation of a semilinear parabolic differential equation with $\mathcal{X}$ as a suitable function space over the spatial domain. In the last case, A corresponds to a linear differential operator and f is a nonlinearity which may involve derivatives of lower order than A.

Inertial manifolds are positively invariant, exponentially attracting, finite dimensional Lipschitz manifolds. The notion goes back to D. Henry and X. Mora [Hen81], [Mor83] and were first introduced and studied by P. Constantin, C. Foias, B. Nicoalenko, G.R. Sell and R. Temam [FST85], [FNST85], [CFNT86] for selfadjoint A. For the construction of inertial manifolds with A being non-selfadjoint see for example [SY92] and [Tem97]. Inertial manifolds are generalizations of center-unstable manifolds and they are more convenient objects which capture the long-time behavior of dynamical systems. If such a manifold exists, then it contains the global attractor $\mathcal{A}$. Usually an inertial manifold $\mathcal{M}$ is seeked as the graph of a sufficiently smooth function m on $P\mathcal{X}$, i.e.

$$\mathcal{M} = \mathrm{graph}(m) := \{x + m(x) : \; x \in P\mathcal{X}\} ,$$

where P is a finite dimensional projector. The finite dimensionality and the exponential attracting property permit the reduction of the dynamics of the infinite or high dimensional equation to the dynamics of a finite or low dimensional ordinary differential equation

$$\dot{x} + Ax = Pf(x + m(x)) \quad \text{in } P\mathcal{X}$$

called inertial form system. A stronger reduction property is the asymptotical completeness property [Rob96]: Each trajectory of the evolution equation tends exponentially to a trajectory in the inertial manifold.

There are a few ways of constructing an inertial manifold. Most of them are generalization of methods developed for the construction of unstable, center-unstable or center manifolds for ordinary differential equations.

The above mentioned notion of inertial manifolds is translated and extended to more general classes of differential equations like nonautonomous differential equations, [GV97], [WF97], [LL99], retarded parabolic differential equations, [TY94], [BdMCR98], or differential equations with random or stochastic perturbations, [Chu95], [BF95], [CL99], [CS01], [DLS01].

The construction of inertial manifolds often is redone for different classes of equations. Our aim is to separate the general structure of the construction from the technical estimates which vary from example to example. So in [KS01] we developed an existence result of inertial manifolds on the abstract level of nonautonomous dynamical systems and we applied it to explicit nonautonomous evolution equations under various assumptions. The main assumptions on the nonautonomous dynamical system are generalizations of the cone invariance property and the squeezing property as geometrical assumptions on the nonautonomous dynamical system. For the proof of the existence result we need some additional technical assumptions on the nonautonomous dynamical system which we called boundedness property and coercivity property. For nonautonomous semilinear evolution equations whose linear part satisfies an exponential dichotomy condition and these properties follows from the global boundedness of the nonlinear part and its global Lipschitz property.

Whereas the global Lipschitz property is a standard assumption and which is also used to verify the cone invariance and squeezing property, in some approaches to inertial manifolds the boundedness assumption is removed or at least replaced by weaker assumptions, for example by the requirement that there is a stationary solution.

For reason of completeness we repeat the essential results of [KS01]. In addition to [KS01], we will introduce another group of technical assumptions which can be verified in the case of evolution equations without boundedness assumption on the nonlinear part but by assuming of an special stationarity property and a quantified coercivity property.

Moreover we give a slight extension to the case the nonautonomous dynamical system acts on a Banach space $\mathcal{X}$ whereas the cone invariance and squeezing property are required only with respect to the weaker norm of a larger space $\mathcal{Y}$. This includes the situation of parabolic evolution equations where the smoothing action of these problems allows to use weaker assumptions on the dynamical system.

2 Nonautonomous Dynamical Systems

2.1 Preliminaries

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space.

Definition 1 (Nonautonomous Dynamical System (NDS)). A *nonautonomous dynamical system* (NDS) on $\mathcal{X}$ is a *cocycle* φ over a *driving system* θ on a set $\mathcal{B}$, i.e.

(i) $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ is a *dynamical system*, i.e. the family $\theta(t, \cdot) = \theta(t) : \mathcal{B} \to \mathcal{B}$ of self-mappings of $\mathcal{B}$ satisfies the *group property*

$$\theta(0) = \mathrm{id}_{\mathcal{B}} , \quad \theta(t+s) = \theta(t) \circ \theta(s)$$

for all $t, s \in \mathbb{R}$.

(ii) $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$ is a *cocycle*, i.e. the family $\varphi(t, b, \cdot) = \varphi(t, b) : \mathcal{X} \to \mathcal{X}$ of self-mappings of $\mathcal{X}$ satisfies the *cocycle property*

$$\varphi(0, b) = \mathrm{id}_{\mathcal{X}} , \quad \varphi(t+s, b) = \varphi(t, \theta(s)b) \circ \varphi(s, b)$$

for all $t, s \geq 0$ and $b \in \mathcal{B}$. Moreover $(t, x) \mapsto \varphi(t, b, x)$ is continuous.

Remark 2. (i) The set $\mathcal{B}$ is called *base* and in applications it has additional structure, e.g. it is a probability space, a topological space or a compact group and the driving system has additional regularity, e.g. it is ergodic or continuous.

(ii) The pair of mappings

$$(\theta, \varphi) : \mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X} \to \mathcal{B} \times \mathcal{X}\,, \quad (t, b, x) \mapsto (\theta(t, b), \varphi(t, b, x))$$

is a special semi-dynamical system a so-called *skew product flow* (usually one requires additionally that (θ, φ) is continuous). If $\mathcal{B} = \{b\}$ consists of one point then the cocycle φ is a semi-dynamical system.

(iii) We use the abbreviations $\theta_t b$ or $\theta(t)b$ for $\theta(t, b)$ and $\varphi(t, b)x$ for $\varphi(t, b, x)$. We also say that φ is an NDS to abbreviate the situation of Definition 1.

Definition 3 (Nonautonomous Set). A family $\mathcal{M} = (\mathcal{M}(b))_{b\in\mathcal{B}}$ of non-empty sets $\mathcal{M}(b) \subset \mathcal{X}$ is called a *nonautonomous set* and $\mathcal{M}(b)$ is called the *b-fiber* of $\mathcal{M}$ or the *fiber* of $\mathcal{M}$ over b. We say that $\mathcal{M}$ is *closed*, *open*, *bounded*, or *compact*, if every fiber has the corresponding property. For notational convenience we use the identification $\mathcal{M} \simeq \{(b, x) :\ b \in \mathcal{B},\ x \in \mathcal{M}(b)\} \subset \mathcal{B} \times \mathcal{X}$.

Definition 4 (Invariance of Nonautonomous Set). A nonautonomous set $\mathcal{M}$ is called *forward invariant* under the NDS φ, if $\varphi(t, b)\mathcal{M}(b) \subset \mathcal{M}(\theta_t b)$ for $t \geq 0$ and $b \in \mathcal{B}$. It is called *invariant*, if $\varphi(t, b)\mathcal{M}(b) = \mathcal{M}(\theta_t b)$ for $t \geq 0$ and $b \in \mathcal{B}$.

Definition 5 (Inertial Manifold). Let φ be an NDS. Then a nonautonomous set $\mathcal{M}$ is called (nonautonomous) *inertial manifold* if

(i) every fiber $\mathcal{M}(b)$ is a *finite-dimensional Lipschitz manifold* in $\mathcal{X}$ of dimension N for an $N \in \mathbb{N}$;

(ii) $\mathcal{M}$ is *invariant*;

(iii) $\mathcal{M}$ is *exponentially attracting*, i.e. there exists a positive constant η such that for every $b \in \mathcal{B}$ and $x \in \mathcal{M}(b)$ there exists an $x' \in \mathcal{M}(b)$ with

$$\|\varphi(t, b)x - \varphi(t, b)x'\|_{\mathcal{X}} \leq K\mathrm{e}^{-\eta t} \qquad \text{for } t \geq 0 \text{ and } b \in \mathcal{B}$$

and a constant $K = K(b, x, x') > 0$.

The property (iii) is also called *exponential tracking property* or *asymptotic completeness property* and x' or $\varphi(\cdot, b)x'$ is said to be the *asymptotic phase* of x or $\varphi(\cdot, b)x$, respectively.

Recall that if $\mathcal{D}$ and $\mathcal{A}$ are nonempty closed sets in $\mathcal{X}$, the Hausdorff semi-metric $d(\mathcal{D}|\mathcal{A})$ is defined by

$$d(\mathcal{D}|\mathcal{A}) := \sup_{x\in\mathcal{D}} d(x, \mathcal{A})\,, \quad d(x, \mathcal{A}) := \inf_{y\in\mathcal{A}} d(x, y) = \inf_{y\in\mathcal{A}} \|x - y\|\,.$$

The appropriate generalization of convergence to a nonautonomous set $\mathcal{A}$ is the *pullback convergence* defined by

$$d(\varphi(t, \theta_{-t}b)x, \mathcal{A}(b)) \to 0 \qquad \text{for } t \to \infty\,,$$

which was introduced in the mid 1990s in the context of random dynamical systems (see Crauel and Flandoli [CF94], Flandoli and Schmalfuss [FS96], and Schmalfuss [Sch92]) and has been used e.g. in numerical dynamics. Note that a similar idea had already been used in the 1960s by Mark Krasnoselski [Kra68] to establish the existence of solutions that exist and remain bounded on the entire time set.

Now we define a handy notion (see Ludwig Arnold [Arn98, Definition 4.1.1(ii)]) excluding exponential growth of a function.

Definition 6 (Temperedness). A function $R : \mathcal{B} \to]0,\infty[$ is called *tempered from above* if for every $b \in \mathcal{B}$

$$\limsup_{t\to\pm\infty} \frac{1}{|t|} \log R(\theta_t b) = 0 \,.$$

Note that the following characterization holds.

Corollary 7. *Suppose that $R : \mathcal{B} \to]0,\infty[$ is a nonautonomous variable. Then the following statements are equivalent:*

(i) R is tempered from above.

(ii) For every $\varepsilon > 0$ and $b \in \mathcal{B}$ there exists a $T > 0$ such that

$$R(\theta_t b) \le \mathrm{e}^{\varepsilon|t|} \qquad \textit{for } |t| \ge T \,.$$

Definition 8 (Nonautonomous Projector). A family $\pi = (\pi(b))_{b\in\mathcal{B}}$ of projectors $\pi(b) \in L(\mathcal{X},\mathcal{X})$ in $\mathcal{X}$ is called *nonautonomous projector*.

(i) π is called *tempered from above* if $b \mapsto \|\pi(b)\|_{L(\mathcal{X},\mathcal{X})}$ is tempered from above.

(ii) π is called N*-dimensional* for an $N \in \mathbb{N}$ if $\dim \mathrm{im}\pi(b) = N$ for every $b \in \mathcal{B}$.

2.2 Inertial Manifolds for Nonautonomous Dynamical Systems

Now let $(\mathcal{X},\mathcal{Y})$ be a pair of two Banach spaces such that $\mathcal{X}$ is continuously embedded in $\mathcal{Y}$,

$$\mathcal{X} \hookrightarrow \mathcal{Y} \,.$$

Our goal is to construct an inertial manifold in $\mathcal{X}$. For this we will use some assumptions with respect to the norm of $\mathcal{X}$. In order to be more general, we will allow that some assumptions are required only with respect to the weaker norm of the larger space $\mathcal{Y}$. To compense the different quality of the norms we need some smoothing action of the dynamical system. Note that in many cases one can use $\mathcal{X} = \mathcal{Y}$, and for a first reading it is good to assume $\mathcal{X} = \mathcal{Y}$ and to overread the technical difficulties dealing with the case $\mathcal{X} \ne \mathcal{Y}$.

Let π_1 be an N-dimensional nonautonomous projector in $\mathcal{Y}$. We define the *complementary projector*

$$\pi_2(b) := \mathrm{id}_{\mathcal{Y}} - \pi_1(b) \qquad \text{for } b \in \mathcal{B} \,.$$

Then

$$\mathcal{X}_1(b) := \pi_1(b)\mathcal{X} \quad \text{and} \quad \mathcal{X}_2(b) := \pi_2(b)\mathcal{X}\,, \qquad b \in \mathcal{B}$$

define nonautonomous sets $\mathcal{X}_i$ consisting of complementary linear subspaces $\mathcal{X}_i(b)$ of $\mathcal{X}$, i.e. $\mathcal{X}_1(b) \oplus \mathcal{X}_2(b) = \mathcal{X}$. For this fact we also write $\mathcal{X}_1 \oplus \mathcal{X}_2 = \mathcal{B} \times \mathcal{X}$. Further let

$$\mathcal{Y}_1(b) := \pi_1(b)\mathcal{Y} \quad \text{and} \quad \mathcal{Y}_2(b) := \pi_2(b)\mathcal{Y}\,, \qquad b \in \mathcal{B}\,.$$

We assume that

$$\mathcal{X}_1(b) = \mathcal{Y}_1(b) \qquad b \in B\,.$$

We say that π_1 is tempered above in $\mathcal{X}$ if the restriction $(\pi_1(b)\big|_{\mathcal{X}})_{b\in\mathcal{B}}$ of π_1 onto $\mathcal{X}$ is tempered above.

We want to construct a nonautonomous inertial manifold

$$\mathcal{M} = (\mathcal{M}(b))_{b\in\mathcal{B}}$$

consisting of manifolds $\mathcal{M}(b)$ which are trivial in the sense that each of them can be described by a single chart, i.e.

$$\mathcal{M}(b) = \operatorname{graph}(m(b,\cdot)) := \{x_1 + m(b,x_1) : \ x_1 \in \mathcal{X}_1(b)\}$$

with $m(b,\cdot) = m(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b)$.

For a positive constant L we introduce the nonautonomous set

$$\mathcal{C}_L := \{(b,x) \in \mathcal{B} \times \mathcal{X} : \ \|\pi_2(b)x\|_{\mathcal{Y}} \le L\|\pi_1(b)x\|_{\mathcal{Y}}\}\,.$$

Since the fibers $\mathcal{C}_L(b)$ are cones it is called *(nonautonomous) cone*. The following definition is a slight generalization of that one in [KS01].

Definition 9 (Cone Invariance). The NDS φ satisfies the (nonautonomous) *cone invariance property* for a cone $\mathcal{C}_L$ if there are a function $\tilde{L} :]0,\infty[\to]0,\infty[$ and a number $T_0 \ge 0$ such that

$$\tilde{L}(t) \le L \qquad \text{for } t \ge T_0$$

and such that for $b \in \mathcal{B}$ and $x, y \in \mathcal{X}$,

$$x - y \in \mathcal{C}_L(b)$$

implies

$$\varphi(t,b)x - \varphi(t,b)y \in \mathcal{C}_{\tilde{L}(t)}(\theta_t b) \qquad \text{for } t > 0\,.$$

Now we define a property of a cocycle φ which describes a kind of squeezing outside a given cone.

Definition 10 (Squeezing Property). The NDS φ satisfies the (nonautonomous) *squeezing property* for a cone $\mathcal{C}_L$ if there exist positive constants K_1, K_2 and η such that for every $b \in \mathcal{B}$, $x, y \in \mathcal{X}$ and $T > 0$ the identity

$$\pi_1(\theta_T b)\varphi(T, b)x = \pi_1(\theta_T b)\varphi(T, b)y$$

implies for all $x' \in \mathcal{X}$ with $\pi_1(b)x' = \pi_1(b)x$ and $x' - y \in \mathcal{C}_L(b)$ the estimates

$$\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)y]\|_{\mathcal{Y}} \le K_i \mathrm{e}^{-\eta t}\|\pi_2(b)[x - x']\|_{\mathcal{Y}}\,, \quad i = 1, 2\,,$$

for $t \in [0, T]$.

Remark 11. The cone invariance and squeezing property are generalization and modifications of the notion of cone invariance and squeezing property for evolution equations. A combination of both properties is sometimes called strong squeezing property, and it was first introduced for the Kuramoto-Sivashinsky equations in [FNST85], [FNST88]. An abstract version of it was developed in [FST89], another formulation of it can be found for example in [Tem88], [FST88], [CFNT89], [Rob93], [JT96]. Essentially, a strong squeezing property states that if the difference of two solutions of the evolution equation belongs to a special cone then it remains in the cone for all further times (that is the cone invariance property); otherwise the distance between the solutions decays exponentially (that is the squeezing property).

Definition 12 (Boundedness Property). The NDS φ satisfies the (nonautonomous) *boundedness property* if for all $t \ge 0$, $b \in \mathcal{B}$ and all $M_1 \ge 0$ there exists a $M_2 \ge 0$ such that for $x \in \mathcal{X}$ with $\|\pi_2(b)x\|_{\mathcal{X}} \le M_1$ the estimate

$$\|\pi_2(\theta_t b)\varphi(t, b)x\|_{\mathcal{X}} \le M_2$$

holds.

Definition 13 (Coercivity Property). The NDS φ satisfies the (nonautonomous) *coercivity property* if for all $t \ge 0$, $b \in \mathcal{B}$ and all $M_3 \ge 0$ there exists an $M_4 \ge 0$ such that for $x \in \mathcal{X}$ with $\|\pi_1(b)x\|_{\mathcal{X}} \ge M_4$ the estimate

$$\|\pi_1(\theta_t b)\varphi(t, b)x\|_{\mathcal{X}} \ge M_3$$

holds.

With the boundedness property we will ensure that the graph transformation mapping can be defined on a complete metric space of bounded functions. The coercivity property will ensure the existence of global homeomorphisms used for the definition of the graph transformation mapping.

Remark 14. As we will show later in Sec. 3.2, for evolution equations the coercivity and boundedness property of φ follows from the boundedness of the nonlinearity and exponential dichotomy properties of the linear part. While a global Lipschitz property of the nonlinearity is used for the cone invariance and squeezing property, too, the boundedness of the nonlinearity is an additional restriction.

Therefore, we introduce now another group of technical assumptions which for evolution equations can be verified without boundedness assumption on the nonlinear part.

Definition 15 (Stationarity Property). The NDS φ satisfies the (nonautonomous) *stationarity property* if there is a uniformly bounded invariant set $\mathcal{I}$.

The stationarity property together with the cone invariance property will allow to define the graph transformation mapping in a space of linearly bounded functions.

Definition 16 (Strong Coercivity Property). The NDS φ satisfies the (nonautonomous) *strong coercivity property with respect to invariant set $\mathcal{I}$ and the cone $\mathcal{C}_L$* if for all $b \in \mathcal{B}$ there exist positive numbers M_5, M_6, M_7 such that for $x \in \mathcal{I}(b) + \mathcal{C}_L(b)$ and all $t \geq 0$ the estimate

$$\|\pi_1(b)x\|_{\mathcal{X}} \leq M_5 \mathrm{e}^{M_6 t}(M_7 + \|\pi_1(\theta_t b)\varphi(t,b)x\|_{\mathcal{X}})$$

holds.

The strong coercivity property will ensure the existence of global homeomorphisms used for the definition of the graph transformation mapping and it will be used to show the contractivity of the graph transformation mapping

Remark 17. As we will show later in Sec. 3.2, for evolution equations the strong coercivity property of φ follows from the uniform boundedness of an invariant set $\mathcal{I}$ and exponential dichotomy properties of the linear part.

If $\mathcal{X} \neq \mathcal{Y}$ we need some properties to compense the weaker norm.

Definition 18 (Smoothing Property). The NDS φ satisfies the *smoothing property* if there are function $M_8, M_9 :]0,\infty[\to]0,\infty[$ such that for $x, y \in \mathcal{X}$, $b \in \mathcal{B}$, and $t > 0$ the Lipschitz estimates

$$\|\varphi(t,b)x - \varphi(t,b)y\|_{\mathcal{X}} \leq M_8(t)\|x-y\|_{\mathcal{Y}}$$

and

$$\|\pi_1(b)[x-y]\|_{\mathcal{Y}} \leq M_9(t)\|\pi_1(\theta_t b)[\varphi(t,b)x - \varphi(t,b)y]\|_{\mathcal{Y}} \qquad \text{if } x-y \in \mathcal{C}_L$$

hold.

Remark 19. For parabolic evolution equations these smoothing property is a consequence of global Lipschitz property of the nonlinearity and the smoothing property of parabolic equations.

Theorem 20 (Existence of Inertial Manifold). *Let φ be an NDS on a Banach space $\mathcal{X} \hookrightarrow \mathcal{Y}$ over a driving system $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$ on a set $\mathcal{B}$ and assume that φ satisfies the cone invariance and squeezing property.*

Moreover let φ satisfy the following technical assumptions:

- *φ possesses the coercivity and boundedness property*

or

- *φ possesses the strong coercivity and stationarity property with respect to the invariant set $\mathcal{I}$ and the cone $\mathcal{C}_L$ with a constant $M_6 < \eta$.*

If $\mathcal{X} \neq \mathcal{Y}$, we further assume that

- *φ possesses the smoothing property, and that π_1 is tempered from above in $\mathcal{X}$, and that there are constants M_{10} and M_{11} with*

$$\|\pi_2(b)\|_{L(\mathcal{X},\mathcal{X})} \leq M_{10}\,, \quad \|\pi_1(b)x\|_{\mathcal{Y}} \leq M_{11}\|\pi_1(b)x\|_{\mathcal{X}} \qquad \text{for } x \in \mathcal{X},\ b \in \mathcal{B}\,.$$

Then there exists an inertial manifold $\mathcal{M} = (\mathcal{M}(b))_{b\in\mathcal{B}}$ of φ with the following properties:

(i) $\mathcal{M}(b)$ is a graph in $\mathcal{X}_1(b) \oplus \mathcal{X}_2(b)$,

$$\mathcal{M}(b) = \{x_1 + m(b, x_1) :\ x_1 \in \mathcal{X}_1(b)\}$$

with a mapping $m(b,\cdot) = m(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b)$ which is globally Lipschitz continuous

$$\|m(b, x_1) - m(b, y_1)\|_{\mathcal{X}} \leq \hat{L}\|x_1 - y_1\|_{\mathcal{X}}$$

with some $\hat{L} \geq 0$, and it satisfies

$$\|m(b, x_1) - m(b, y_1)\|_{\mathcal{Y}} \leq L\|x_1 - y_1\|_{\mathcal{Y}}$$

with L from the cone invariance property.

(ii) $\mathcal{M}$ is exponentially attracting in $\mathcal{X}$

$$\|\varphi(t, b)x - \varphi(t, b)x'\|_{\mathcal{X}} \leq \hat{K}\mathrm{e}^{-\eta t}\|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{X}}$$

for $t \geq 1$, $i = 1, 2$ with an asymptotic phase $x' = x'(b, x) \in \mathcal{M}(b)$ of x and some $\hat{K}$ independent of x, x', b, t, and we have

$$\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)x']\|_{\mathcal{Y}} \leq K_i\mathrm{e}^{-\eta t}\|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{Y}}$$

for $t \geq 0$, $i = 1, 2$ with $K_1, K_2 > 0$ from the squeezing property.

(iii) If in addition π_1 is tempered from above in $\mathcal{X}$, then $\mathcal{M}$ is pullback attracting in $\mathcal{X}$, more precisely, there is a $T \geq 0$ such that for each bounded set $\mathcal{D} \subset \mathcal{X}$

$$d(\varphi(t, \theta_{-t}b)\mathcal{D}|\mathcal{M}(b)) \leq \mathrm{e}^{-\eta t/2}d(\mathcal{D}|\mathcal{M}(\theta_{-t}b)) \qquad \text{for } t \geq T\,. \tag{1}$$

Proof. We divide the proof into two parts. In the first part we assume that φ possesses the boundedness and coercivity property. Since the proof is rather involved we split this part into four steps. In the first step we define the graph transformation mapping. In the second step we show that it has a unique fixed point $m(b)$ which gives rise to a nonautonomous invariant set $\mathcal{M}$ of Lipschitz manifolds $\mathcal{M}(b)$. In the third step the exponential tracking property is proved and in the fourth step we investigate the pullback attractivity of $\mathcal{M}$.

In the second part we assume that φ satisfies the strong coercivity and stationarity property. There we will repeat only these parts of the proof which are changing.

Part I: Let φ satisfy the boundedness and coercivity property.

Step 1: Definition of graph transformation mapping

We construct the manifolds $\mathcal{M}(b) = \operatorname{graph}(m(b))$ as the fixed point of the cocycle φ acting on a certain class of functions g with

$$g(b,\cdot) = g(b) : \mathcal{X}_1(b) \to \mathcal{X}_2(b)\,, \qquad b \in \mathcal{B}\,.$$

The set $\mathcal{G}$ of mappings of the form

$$\mathcal{X}_1 \ni (b, x_1) \mapsto (b, g(b, x_1)) \in \mathcal{X}_2\,,$$

such that g is bounded and $g(b,\cdot)$ is continuous for every $b \in \mathcal{B}$ is a Banach space with the norm

$$\|g\|_{\mathcal{G}} = \sup_{(b,x_1)\in\mathcal{X}_1} \|g(b,x_1)\|_{\mathcal{X}}\,.$$

Moreover let $\mathcal{G}_L$ denote the subset of $\mathcal{G}$ containing all mappings which satisfy the global Lipschitz condition

$$\|g(b,x_1) - g(b,y_1)\|_{\mathcal{Y}} \le L\|x_1 - y_1\|_{\mathcal{Y}}$$

for $(b,x_1),(b,y_1) \in \mathcal{X}_1$ with L from the cone invariance property. Note that both $\mathcal{G}$ and $\mathcal{G}_L \subset \mathcal{G}$ are complete metric spaces.

Let $T > 0$ be arbitrary, but fixed. We wish to define the *graph transformation mapping* $G^T : \mathcal{G}_L \to \mathcal{G}$ such that

$$\operatorname{graph}((G^T g)(\theta_T b,\cdot)) = \varphi(T,b)\operatorname{graph}(g(b,\cdot))$$

and this means that $\tilde{x} \in \operatorname{graph}((G^T g)(\theta_T b,\cdot))$ equals $\varphi(T,b)x$ for some $x \in \operatorname{graph}(g(b,\cdot))$. More precisely, for an $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$ we want to define

$$(G^T g)(\theta_T b, \tilde{x}_1) = \tilde{x}_2 \in \mathcal{X}_2(\theta_T b)$$

by determining an $x \in \operatorname{graph}(g(b,\cdot))$ such that

$$\pi_1(\theta_T b)\varphi(T,b)x = \tilde{x}_1 \quad \text{and} \quad \pi_2(\theta_T b)\varphi(T,b)x =: (G^T g)(\theta_T b, \tilde{x}_1)\,.$$

To this end we show that for each $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, $g \in \mathcal{G}_L$, the boundary value problem

$$x \in \operatorname{graph}(g(b,\cdot))\,, \quad \pi_1(\theta_T b)\varphi(T,b)x = \tilde{x}_1 \tag{2}$$

has a unique solution $x = \beta(T, b, \tilde{x}_1, g)$.

Uniqueness of a solution of (2). Assume there exist x and y with

$$x, y \in \operatorname{graph}(g(b,\cdot))\,, \quad \pi_1(\theta_T b)\varphi(T,b)x = \pi_1(\theta_T b)\varphi(T,b)y = \tilde{x}_1\,.$$

We get $x-y \in \mathcal{C}_L(b)$ and the squeezing property (with $x' = x$) implies the identity

$$\varphi(t,b)x = \varphi(t,b)y \qquad \text{for } t \in [0,T]\,.$$

Hence $x = y$ and there is at most one solution $x = \beta(T, b, \tilde{x}_1, g)$ of (2).

Existence of a solution of (2). Note that by assumption $\mathcal{X}_1(b) = \mathcal{Y}_1(b)$ for $b \in \mathcal{B}$. Let $T > 0$, $b \in \mathcal{B}$, $g \in \mathcal{G}_L$ be fixed and define $H\colon \mathcal{X}_1(b) \to \mathcal{X}_1(\theta_T b)$ by

$$H(x_1) := \pi_1(\theta_T b)\varphi(T,b)(x_1 + g(b,x_1))\,.$$

By the continuity of $\varphi(T,b)$ and $g(b,\cdot)$, H is continuous. For $\tilde{x}_1 \in H\mathcal{X}_1(b) \subset \mathcal{X}_1(\theta_T b)$, any x_1 in the preimage $H^{-1}(\tilde{x}_1) = \{x_1 \in \mathcal{X}_1(b) : \; H(x_1) = \tilde{x}_1\}$ gives rise to a solution $x = x_1 + g(b, x_1)$ of the boundary value problem (2). As we have already seen, there exists at most one solution denoted by $\beta(T, b, \tilde{x}_1, g)$ and therefore the inverse H^{-1} of H is given by

$$H^{-1}(\tilde{x}_1) = \pi_1(b)\beta(T,b,\tilde{x}_1,g) \qquad \text{on} \quad H\mathcal{X}_1(\theta_T b)\,.$$

Now we show the continuity of $H^{-1} : H\mathcal{X}_1(b) \to \mathcal{X}_1(b)$. Suppose, there is a $\tilde{\xi} \in H\mathcal{X}_1(b) \subset \mathcal{X}_1(\theta_T b)$ such that H^{-1} is not continuous at $\tilde{\xi}$. Then there are $\varepsilon > 0$ and a sequence $(\tilde{\xi}_k)_{k\in\mathbb{N}}$ in $\mathcal{X}_1(\theta_T b)$ such that $\tilde{\xi}_k \to \tilde{\xi}_0$ as $k \to \infty$ and

$$\|\xi_k - \xi_0\|_{\mathcal{X}} \geq \varepsilon \qquad \text{for all } k \in \mathbb{N} \tag{3}$$

where $\xi_k := \pi_1(b)\beta(T, b, \tilde{\xi}_k, g)$ for $k = 0, 1, \ldots$.

First, we suppose that there is a subsequence of $(\xi_k)_{k\in\mathbb{N}}$, denoted for shortness again by $(\xi_k)_{k\in\mathbb{N}}$, with $\|\xi_k\|_{\mathcal{X}} \to \infty$ as $k \to \infty$. Then the coercivity property of φ implies

$$\|\tilde{\xi}_k\|_{\mathcal{X}} = \|H(\xi_k)\|_{\mathcal{X}} = \|\pi_1(\theta_T b)\varphi(T,b)(\xi_k + g(b,\xi_k))\|_{\mathcal{X}} \to \infty \qquad \text{as } k \to \infty\,,$$

but this contradicts $\tilde{\xi}_k \to \tilde{\xi}_0$.

Therefore we have proved that $(\xi_k)_{k\in\mathbb{N}}$ is bounded. Since $\mathcal{X}_1(b)$ is a finite-dimensional space, there is a convergent subsequence, denoted for shortness again by $(\xi_k)_{k\in\mathbb{N}}$, with a limit

$$\xi_\infty = \lim_{k\to\infty} \xi_k \in \mathcal{X}_1(b)\,. \tag{4}$$

By the continuity of H, we have $H(\xi_k) \to H(\xi_\infty)$. Since also $H(\xi_k) = \tilde{\xi}_k \to \tilde{\xi}_0 = H(\xi_0)$ we get $\xi_0 = \xi_\infty$ in contradiction to (3) and (4). Therefore, H and H^{-1} are continuous.

Now we show that H is onto, i.e. satisfies

$$H\mathcal{X}_1(b) = \mathcal{X}_1(\theta_T b)\,. \tag{5}$$

By the coercivity of φ, we have the norm coercivity $\|H(\xi)\|_{\mathcal{X}} \to \infty$ for $\|\xi\|_{\mathcal{X}} \to \infty$ of H. Since $\mathcal{X}_1(b)$ is finite-dimensional, H is a sequentially compact mapping. By [Rhe69, Theorem 3.7], H is a homeomorphism from $\mathcal{X}_1(b)$ onto $\mathcal{X}_1(\theta_T b)$ and hence we have (5).

Thus we have unique solvability of (2), and we can define the graph transformation mapping G^T by

$$(G^T g)(\theta_T b, \tilde{x}_1) = \pi_2(\theta_T b)\varphi(T, b)\beta(T, b, \tilde{x}_1, g)$$

for $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, and $g \in \mathcal{G}_L$. Note that we have

$$\operatorname{graph}((G^T g)(\theta_T b, \cdot)) = \varphi(T, b)\operatorname{graph}(g(b, \cdot)) .$$

Since $H^{-1} : \mathcal{X}_1(\theta_T b) \to \mathcal{X}_1(b)$, $g(b, \cdot) : \mathcal{X}_1(b) \to \mathcal{X}_2(b)$ and $\pi_1(b)\varphi(T, b) : \mathcal{X} \to \mathcal{X}$ are continuous, and

$$\beta(T, b, \tilde{x}_1, g) = H^{-1}(\tilde{x}_1) + g(b, H^{-1}(\tilde{x}_1))$$

holds, the mapping $(G^T g)(\theta_T b, \cdot) : \mathcal{X}_1(\theta_T b) \to \mathcal{X}_2(\theta_T b)$ is also continuous.

Now we show that

$$\|G^T g\|_{\mathcal{G}} < \infty .$$

Since $g \in \mathcal{G}$ there is a M_1 with $\|\pi_2(b)x\|_{\mathcal{X}} \leq M_1$ for all $b \in \mathcal{B}$ and all $x \in \operatorname{graph}(g(b, \cdot))$. By the boundedness property of φ there is a M_2 such that

$$\|\pi_2(\theta_T b)\varphi(t, b)x\|_{\mathcal{X}} \leq M_2$$

for all $b \in \mathcal{B}$ and all $x \in \mathcal{X}$ with $\|\pi_2(b)x\|_{\mathcal{X}} \leq M_1$. Let $b \in \mathcal{B}_0$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$ be arbitrary. Then $\beta(T, b, \tilde{x}_1, g) \in \operatorname{graph}(g(b, \cdot))$, hence $\|\pi_2(b)\beta(T, b, \tilde{x}_1, g)\|_{\mathcal{X}} \leq M_1$, and therefore

$$\|(G^T g)(\theta_T b, \tilde{x}_1)\|_{\mathcal{X}} = \|\pi_2(\theta_T b)\varphi(T, b)\beta(T, b, \tilde{x}_1, g)\|_{\mathcal{X}} \leq M_2$$

proving that $\|G^T g\|_{\mathcal{G}} < \infty$, i.e. $G^T g \in \mathcal{G}$.

Let $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1, \tilde{x}_2 \in \mathcal{X}_1(\theta_T b)$, $g \in \mathcal{G}_L$ be arbitrary. Since $\beta(T, b, \tilde{x}_1, g)$, $\beta(T, b, \tilde{x}_2, g) \in \operatorname{graph}(g(b, \cdot))$, we get

$$\beta(T, b, \tilde{x}_1, g) - \beta(T, b, \tilde{x}_2, g) \in \mathcal{C}_L(b) \tag{6}$$

and the cone invariance property implies for $T \geq T_0$ a Lipschitz estimate for $G^T g$,

$$\begin{aligned}
&\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T g)(\theta_T b, \tilde{x}_2)\|_{\mathcal{Y}} \\
&\quad \leq L\|\pi_1(\theta_T b)(\varphi(T, b)\beta(T, b, \tilde{x}_1, g) - \varphi(T, b)\beta(T, b, \tilde{x}_2, g))\|_{\mathcal{Y}} \\
&\quad = L\|\tilde{x}_1 - \tilde{x}_2\|_{\mathcal{Y}} ,
\end{aligned}$$

i.e. $(G^T g)(\theta_T b, \cdot)$ satisfies a Lipschitz condition as mapping from $\mathcal{Y}_1(b)$ into $\mathcal{Y}$ with Lipschitz constant L. Thus G^T maps $\mathcal{G}_L$ into $\mathcal{G}$ for every $T \geq 0$, and it is self-mapping for $T \geq T_0$.

Moreover, using the smoothing property, for $T > 0$ we obtain

$$\begin{aligned}
&\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T g)(\theta_T b, \tilde{x}_2)\|_{\mathcal{X}} \\
&\quad \leq \|\pi_2(\theta_T b)\|_{L(\mathcal{X},\mathcal{X})}\|\varphi(T, b)\beta(T, b, \tilde{x}_1, g) - \varphi(T, b)\beta(T, b, \tilde{x}_2, g)\|_{\mathcal{X}} \\
&\quad \leq M_{10}M_8(T)\|\beta(T, b, \tilde{x}_1, g) - \beta(T, b, \tilde{x}_2, g)\|_{\mathcal{Y}} \\
&\quad \leq (1 + L)M_8(T)M_{10}\|\pi_1(b)[\beta(T, b, \tilde{x}_1, g) - \beta(T, b, \tilde{x}_2, g)]\|_{\mathcal{Y}} \\
&\quad \leq (1 + L)M_8(T)M_9(T)M_{10}\|\tilde{x}_1 - \tilde{x}_2\|_{\mathcal{Y}} \\
&\quad \leq \hat{L}\|\tilde{x}_1 - \tilde{x}_2\|_{\mathcal{X}}
\end{aligned}$$

where

$$\hat{L} := (1+L)M_8(T)M_9(T)M_{10}M_{11} \, .$$

Step 2: Unique fixed-point of graph transformation mappings

Let $T \geq 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, $g, h \in \mathcal{G}$, and $x \in \operatorname{graph}(g(b,\cdot))$, $y \in \operatorname{graph}(h(b,\cdot))$ with

$$\pi_1(\theta_T b)\varphi(T,b)x = \pi_1(\theta_T b)\varphi(T,b)y = \tilde{x}_1 \, .$$

Define

$$x' := \pi_1(b)x + h(b, \pi_1(b)x) \, .$$

Then $x' - y \in \mathcal{C}_L$, and the squeezing property implies

$$\|\pi_i(\theta_t b)[\varphi(t,b)x - \varphi(t,b)y]\|_{\mathcal{Y}} \leq K_2 \mathrm{e}^{-\eta t}\|g(b,\pi_1(b)x) - h(b,\pi_1(b)x)\|_{\mathcal{Y}}$$

for $t \in [0,T]$. If $\mathcal{X} = \mathcal{Y}$ we obtain

$$\begin{aligned}
&\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T h)(\theta_T b, \tilde{x}_1)\|_{\mathcal{X}} \\
&\qquad \leq K_2 \mathrm{e}^{-\eta T}\|g(b, \pi_1(b)\beta(T,b,\tilde{x}_1,g)) - h(b,\pi_1(b)\beta(T,b,\tilde{x}_1,g))\|_{\mathcal{X}} \, ,
\end{aligned}$$

and passing to the sup over all $(\theta_T b, \tilde{x}_1) \in \mathcal{X}_1$ we get

$$\|G^T g - G^T h\|_{\mathcal{G}} \leq \kappa(T)\|g - h\|_{\mathcal{G}} \tag{7}$$

for all $T > 0$, $g, h \in \mathcal{G}_L$, where

$$\kappa(T) := K_2 \mathrm{e}^{-\eta T} \, .$$

If $\mathcal{X} \neq \mathcal{Y}$ we proceed as follows: Because of

$$(G^T g)(\theta_T b, \tilde{x}_i) = \pi_2(\theta_T b)\varphi(1, \theta_{T-1}b)\varphi(T-1,b)\beta(T,b,\tilde{x}_i,g)$$

and using the smoothing property and the continuous embedding of $\mathcal{X}$ in $\mathcal{Y}$ we find

$$\begin{aligned}
&\|(G^T g)(\theta_T b, \tilde{x}_i) - (G^T g)(\theta_T b, \tilde{x}_i)\|_{\mathcal{X}} \\
&\qquad \leq M_{10}M_8(1)\|\varphi(T-1,b)\beta(T,b,\tilde{x}_1,g) - \varphi(T-1,b)\beta(T,b,\tilde{x}_2,g)\|_{\mathcal{Y}} \\
&\qquad \leq \kappa(T)\|g(b,\pi_1(b)\beta(T,b,\tilde{x}_1,g)) - h(b,\pi_1(b)\beta(T,b,\tilde{x}_1,g))\|_{\mathcal{X}}
\end{aligned}$$

and hence (7) with

$$\kappa(T) = (K_1 + K_2)M_8(1)M_{10}C\mathrm{e}^{-\eta(T-1)}$$

for $T > 1$, where C is an embedding constant for the embedding from $\mathcal{X}$ into $\mathcal{Y}$.

Since $\eta > 0$ and since π_2 is tempered from above in $\mathcal{X}$, there is a positive $T_1 \geq T_0$ with $\kappa(T) < 1$ for $T \geq T_1$. Thus, for $T \geq T_1$, G^T is a contractive self-mapping on the complete metric space $\mathcal{G}_L$. Now choose and fix an arbitrary $\tilde{T} \geq T_1$

and let m denote the unique fixed-point of $G^{\tilde{T}}$ in $\mathcal{G}_L$. We show that m is the unique fixed-point of G^T for every $T \geq 0$.

For every $T \geq 0$ the mapping $G^T m \in \mathcal{G}$ is uniquely determined by the graphs

$$\operatorname{graph}((G^T m)(\theta_T b, \cdot)) = \varphi(T, b)\operatorname{graph}(m(b, \cdot)) , \qquad b \in \mathcal{B} .$$

For $T \geq 0$ and $b \in \mathcal{B}$ we have the identity

$$\begin{aligned}
\operatorname{graph}((G^{T+\tilde{T}} m)(\theta_{T+\tilde{T}} b, \cdot)) &= \varphi(T + \tilde{T}, b)\operatorname{graph}(m(b, \cdot)) \\
&= \varphi(T, \theta_{\tilde{T}} b)\varphi(\tilde{T}, b)\operatorname{graph}(m(b, \cdot)) \\
&= \varphi(T, \theta_{\tilde{T}} b)\operatorname{graph}(m(\theta_{\tilde{T}} b, \cdot)) \\
&= \operatorname{graph}((G^T m)(\theta_{T+\tilde{T}} b, \cdot))
\end{aligned}$$

and therefore $G^T m = G^{T+\tilde{T}} m \in \mathcal{G}_L$. Hence the composition $G^T G^{T'} m$ makes sense for $T, T' \geq 0$ and we get

$$\begin{aligned}
\operatorname{graph}((G^T G^{T'} m)(\theta_{T+T'} b, \cdot)) &= \varphi(T, \theta_{T'} b)\operatorname{graph}(G^{T'} m(\theta_{T'} b, \cdot)) \\
&= \varphi(T, \theta_{T'} b)\varphi(T', b)\operatorname{graph}(m(b, \cdot)) \\
&= \varphi(T + T', b)\operatorname{graph}(m(b, \cdot)) \\
&= \operatorname{graph}((G^{T+T'} m)(\theta_{T+T'} b, \cdot))
\end{aligned}$$

and therefore $G^T G^{T'} m = G^{T'} G^T m = G^{T+T'} m$ for $T, T' \geq 0$. We get

$$G^{\tilde{T}}(G^T m) = G^T(G^{\tilde{T}} m) = G^T m .$$

Thus $G^T m$ equals the unique fixed-point m of $G^{\tilde{T}}$ and we have

$$G^T m = m \qquad \text{for } T \geq 0 .$$

To prove the uniqueness of the fixed-point m of G^T, assume that m^* is another fixed-point. But then m and m^* are both fixed-points of G^{kT} for every $k \in \mathbb{N}$. Choosing k large enough such that $kT \geq T_1$ we know that G^{kT} has a unique fixed-point and this implies $m = m^*$.

Thus m is the unique mapping in $\mathcal{G}_L$ with the invariance property

$$\varphi(t, b)\operatorname{graph}(m(b, \cdot)) = \operatorname{graph}(m(\theta_t b, \cdot)) \qquad \text{for } t \geq 0 \text{ and } b \in \mathcal{B} .$$

We define $\mathcal{M}(b) := \operatorname{graph}(m(b, \cdot))$ for $b \in \mathcal{B}$.

Step 3: Existence of asymptotic phases

Let $b \in \mathcal{B}$ and $x \in \mathcal{X}$ be arbitrary and let $(t_k)_{k\in\mathbb{N}}$ be a monotonously increasing sequence of positive real numbers t_k with $t_k \to \infty$ for $k \to \infty$. Define $y' := \pi_1(b)x + m(b, \pi_1(b)x) \in \operatorname{graph}(m(b, \cdot))$ and

$$x_k := \beta(t_k, b, \pi_1(\theta_{t_k} b)\varphi(t_k, b)x, m) \in \operatorname{graph}(m(b, \cdot)) .$$

We get $y' - x_k \in \mathcal{C}_L(b)$ and the squeezing property implies for $i = 1, 2$, $t \in [0, t_k]$

$$\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)x_k]\|_{\mathcal{Y}} \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{Y}} \, .$$

In particular, we find for $i = 1$ and $t = 0$

$$\begin{aligned} \|\pi_1(b)x_k\|_{\mathcal{Y}} &\leq \|\pi_1(b)x\|_{\mathcal{Y}} + \|\pi_1(b)[x - x_k]\|_{\mathcal{Y}} \\ &\leq \|\pi_1(b)x\|_{\mathcal{Y}} + K_1 \|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{Y}} \, . \end{aligned}$$

Therefore we have proved that $(\pi_1(b)x_k)_{k\in\mathbb{N}} \subset \mathcal{X}_1(b) = \mathcal{Y}_1(b)$ is bounded. Since $\mathcal{X}_1(b)$ is a finite-dimensional space, there is a convergent subsequence, denoted again by $(\pi_1(b)x_k)_{k\in\mathbb{N}}$. Since

$$x_k = \pi_1(b)x_k + m(b, \pi_1(b)x_k)$$

and $m(b, \cdot)$ is continuous, also $(x_k)_{k\in\mathbb{N}}$ is converging to some

$$x' \in \operatorname{graph}(m(b, \cdot)) \, .$$

Then we get for $i = 1, 2$

$$\begin{aligned} &\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)x']\|_{\mathcal{Y}} \\ &\quad \leq \|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)x_k]\|_{\mathcal{Y}} + \|\pi_i(\theta_t b)[\varphi(t, b)x_k - \varphi(t, b)x']\|_{\mathcal{Y}} \\ &\quad \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{Y}} + \|\pi_i(\theta_t b)[\varphi(t, b)x_k - \varphi(t, b)x']\|_{\mathcal{Y}} \end{aligned}$$

for all $T > 0$, $t \in [0, T]$ and all $k \in \mathbb{N}_{>0}$ with $t_k \geq T$. By the continuity of $\varphi(t, b)$, and because of $x_k \to x'$, the second term can be made arbitrary small for each fixed $t \in [0, T]$ choosing k large enough. Therefore,

$$\|\pi_i(\theta_t b)[\varphi(t, b)x - \varphi(t, b)x']\|_{\mathcal{Y}} \leq K_i \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{Y}}$$

for $t \geq 0$, i.e. $x' \in \mathcal{M}(b)$ is an asymptotic phase of x if $\mathcal{X} = \mathcal{Y}$.

If $\mathcal{X} \neq \mathcal{Y}$, then we note that the smoothing property and the continuous embedding of $\mathcal{X}$ in $\mathcal{Y}$ implies the existence of a constant $\hat{K}$ with

$$\|\varphi(t, b)x - \varphi(t, b)x'\|_{\mathcal{X}} \leq \hat{K} \mathrm{e}^{-\eta t} \|\pi_2(b)x - m(b, \pi_1(b)x)\|_{\mathcal{X}} \qquad \text{for } t \geq 1 \, .$$

Step 4: Pullback attractivity

Note that with π_1 also the complementary projector π_2 is tempered from above. Since $\varphi(t, b)x' \in \mathcal{M}(\theta_t b)$ for every $x \in \mathcal{X}$, $t \geq 0$, $b \in \mathcal{B}$. Step 3 implies

$$d(\varphi(t, b)x, \mathcal{M}(\theta_t b)) \leq \hat{K} \mathrm{e}^{-\eta t} \|\pi_2(b)[x - x']\|_{\mathcal{X}} \qquad \text{for } t \geq 1$$

with $x' = \pi_1(b)x + m(b, \pi_1(b)x)$ and some constant $\hat{K}$. Let $z \in \mathcal{M}(b)$ be arbitrary. Then, because of $x' \in \mathcal{M}(b)$, the Lipschitz property of m, and $\pi_1(b)x = \pi_1(b)x'$,

$$\begin{aligned} \|\pi_2(b)[x - x']\|_{\mathcal{X}} &\leq \|\pi_2(b)[x - z]\|_{\mathcal{X}} + \hat{L} \|\pi_2(b)[z - x']\|_{\mathcal{X}} \\ &= \left(\|\pi_2(b)\|_{L(\mathcal{X},\mathcal{X})} + \hat{L} \|\pi_1(b)\|_{L(\mathcal{X},\mathcal{X})} \right) \|x - z\|_{\mathcal{X}} \, . \end{aligned}$$

Hence

$$\|\pi_2(b)[x - x']\|_{\mathcal{X}} \le \Big(\|\pi_2(b)\|_{L(\mathcal{X},\mathcal{X})} + \hat{L}\|\pi_1(b)\|_{L(\mathcal{X},\mathcal{X})}\Big)\, d(x, \mathcal{M}(b)) \,.$$

Replacing b by $\theta_{-t}b$, choosing a $T > 1$ such that

$$\tilde{K} \cdot \big(\|\pi_2(\theta_{-t}b)\|_{L(\mathcal{X},\mathcal{X})} + L\|\pi_1(\theta_{-t}b)\|_{L(\mathcal{X},\mathcal{X})}\big) \le \mathrm{e}^{\frac{\eta}{2}t} \qquad \text{for } t \ge T$$

(Corollary 7), we obtain

$$d(\varphi(t, \theta_{-t}b)x, \mathcal{M}(b)) \le \mathrm{e}^{-\frac{\eta}{2}t} d(x, \mathcal{M}(\theta_{-t}b)) \le \mathrm{e}^{-\frac{\eta}{2}t} d(\mathcal{D}, \mathcal{M}(\theta_{-t}b))$$

for $t \ge T$ proving the pullback attractivity of $\mathcal{M}$ with T independent of $\mathcal{D}$.

Part II: Let φ satisfy the stationarity property and strong coercivity property with respect to the invariant set $\mathcal{I}$ and the cone $\mathcal{C}_L$ with a constant $M_6 < \eta$.

The main difference in comparison to Part I concerns the utilization of another complete metric space (a space of linearly bounded functions instead of a space of bounded functions) and therefore an appropriate proof of contraction property of the graph transformation mapping.

Let $\mathcal{G}$ be the set mappings of the form

$$\mathcal{X}_1 \ni (b, x_1) \mapsto (b, g(b, x_1)) \in \mathcal{X}_2 \,,$$

such that $x_1 + g(b, x_1) \in \mathcal{I}(b) + C_L(b)$ for every $(b, x_1) \in \mathcal{X}_1$ and that $g(b, \cdot)$ is linearly bounded uniformly in b. i.e., for g there are $C_1, C_2 \ge 0$ with

$$\|g(b, x_1)\|_{\mathcal{X}} \le C_0 + C_1\|x_1\|_{\mathcal{Y}} \qquad \text{for } (b, x_1) \in \mathcal{X}_1 = \mathcal{Y}_1 \,.$$

We equipped $\mathcal{G}$ with the norm

$$\|g\|_{\mathcal{G}} = \sup_{(b,x_1)\in\mathcal{X}_1} \frac{\|g(b, x_1)\|_{\mathcal{X}}}{M_7 + \|x_1\|_{\mathcal{Y}}} \,.$$

As in the proof of Theorem 20, let $\mathcal{G}_L$ denote the subset of $\mathcal{G}$ containing all mappings which satisfy the global Lipschitz condition

$$\|g(b, x_1) - g(b, y_1)\|_{\mathcal{Y}} \le L\|x_1 - y_1\|_{\mathcal{Y}}$$

for $(b, x_1), (b, y_1) \in \mathcal{X}_1$ with L from the cone invariance property. Note that both $\mathcal{G}$ and $\mathcal{G}_L \subset \mathcal{G}$ are complete metric spaces.

As in the proof of Theorem 20, the strong coercivity property and the squeezing property allow to define the graph transformation mapping $G^T : \mathcal{G}_L \to \mathcal{G}$ for $T > 0$ by

$$(G^T g)(\theta_T b, \tilde{x}_1) = \pi_2(\theta_T b)\varphi(T, b)\beta(T, b, \tilde{x}_1, g)$$

for $T > 0$, $b \in \mathcal{B}$, $\tilde{x}_1 \in \mathcal{X}_1(\theta_T b)$, and $g \in \mathcal{G}_L$, where $x = \beta(T, b, \tilde{x}_1, g) \in \mathcal{I}(b) + \mathcal{C}_L(b)$ is the unique solution of the boundary value problem

$$x \in \operatorname{graph}(g(b, \cdot)) \,, \quad \pi_1(\theta_T b)\varphi(T, b)x = \tilde{x}_1 \,. \tag{8}$$

The cone invariance property implies

$$\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T g)(\theta_T b, \tilde{x}_2)\|_{\mathcal{X}} \leq \tilde{L}(T)\|\tilde{x}_1 - \tilde{x}_2\|_{\mathcal{X}}$$

for $T \geq T_0$, $b \in \mathcal{B}$, $\tilde{x}_1, \tilde{x}_2 \in \mathcal{X}_1(\theta_T b)$, $g \in \mathcal{G}_L$. Especially, by the stationarity property for each $x \in \mathcal{I}(b)$ with $\pi_2(b)x = g(b, \pi_1(b)x)$ we have $\tilde{x}_2 + (G^T g)(\theta_T b, \tilde{x}_2) \in \mathcal{I}(\theta_T b)$ for $\tilde{x}_2 = \pi_1(\theta_T b)x$. Therefore

$$\|(G^T g)(b, x_1) - (G^T g)(b, x_2)\|_{\mathcal{Y}} \leq L\|x_1 - x_2\|_{\mathcal{Y}}$$

and

$$(G^T g)(b, x_1) \in \mathcal{I}(b) + \mathcal{C}_L$$

for $T \geq T_0$, $b \in \mathcal{B}$, $x_1, x_2 \in \mathcal{X}_1(b)$, $g \in \mathcal{G}_L$. As in Step 1 of Part I of the proof follows that $G^T g$ is a Lipschitz mapping from $\mathcal{X}_1$ into $\mathcal{X}_2$. Thus, G^T maps $\mathcal{G}_L$ into itself for $T \geq T_0$.

Remains to show the contractivity of G^T with respect to the norm $\|\cdot\|_{\mathcal{G}}$ of the new space $\mathcal{G}$. Proceeding as in Step 2 of Part I of the proof, the squeezing property implies

$$\|\pi_2(\theta_t b)[\varphi(t, b)x - \varphi(t, b)y]\|_{\mathcal{Y}} \leq K_2 \mathrm{e}^{-\eta t}\|g(b, \pi_1(b)x) - h(b, \pi_1(b)x)\|_{\mathcal{Y}}$$

for $t \in [0, T]$. We restrict us to the more complicate case $\mathcal{X} \neq \mathcal{Y}$. Because of

$$(G^T g)(\theta_T b, \tilde{x}_i) = \pi_2(\theta_T b)\varphi(1, \theta_{T-1}b)\varphi(T-1, b)\beta(T, b, \tilde{x}_i, g)$$

and using the smoothing property and the continuous embedding of $\mathcal{X}$ in $\mathcal{Y}$ we find

$$\begin{aligned}
&\|(G^T g)(\theta_T b, \tilde{x}_i) - (G^T g)(\theta_T b, \tilde{x}_i)\|_{\mathcal{X}} \\
&\quad \leq M_{10} M_8(1)\|\varphi(T-1, b)\beta(T, b, \tilde{x}_1, g) - \varphi(T-1, b)\beta(T, b, \tilde{x}_2, g)\|_{\mathcal{Y}} \\
&\quad \leq \tilde{\kappa}(T)\|g(b, \pi_1(b)\beta(T, b, \tilde{x}_1, g)) - h(b, \pi_1(b)\beta(T, b, \tilde{x}_1, g))\|_{\mathcal{X}}
\end{aligned}$$

with

$$\tilde{\kappa}(T) = (K_1 + K_2) M_8(1) M_{10} C \mathrm{e}^{-\eta(T-1)}$$

for $T > 1$ where C is an embedding constant for the embedding from $\mathcal{X}$ into $\mathcal{Y}$.

Thus

$$\begin{aligned}
&\frac{\|(G^T g)(\theta_T b, \tilde{x}_1) - (G^T h)(\theta_T b, \tilde{x}_1)\|_{\mathcal{X}}}{M_7 + \|\pi_1(\theta_T b)\tilde{x}_1\|_{\mathcal{X}}} \\
&\qquad \leq \tilde{\kappa}(T) k_5 \frac{\|g(b, \pi_1(b)\beta(T, b, \tilde{x}_1, g)) - h(b, \pi_1(b)\beta(T, b, \tilde{x}_1, g))\|_{\mathcal{X}}}{M_7 + \|\pi_1(b)\beta(T, b, \tilde{x}_1, g)\|_{\mathcal{X}}}
\end{aligned}$$

with

$$k_5 := \frac{M_7 + \|\pi_1(b)\beta(T, b, \tilde{x}_1, g)\|_{\mathcal{X}}}{M_7 + \|\pi_1(\theta_T b)\tilde{x}_1\|_{\mathcal{X}}}\,.$$

Using the strong coercivity property we find

$$k_5 \leq \frac{M_7 + M_5 \mathrm{e}^{M_6 T}(M_7 + \|\pi_1(\theta_T b)\tilde{x}_1\|_{\mathcal{X}})}{M_7 + \|\pi_1(\theta_T b)\tilde{x}_1\|_{\mathcal{X}}} \leq 1 + M_5 \mathrm{e}^{M_6 T}$$

and passing to the sup over all $(\theta_T b, \tilde{x}_1) \in \mathcal{X}_1$ we get

$$\|G^T g - G^T h\|_{\mathcal{G}} \leq \kappa(T) \|g - h\|_{\mathcal{G}}$$

for all $T > 1$, $g, h \in \mathcal{G}_L$, where

$$\begin{aligned} \kappa(T) &:= (1 + M_5 \mathrm{e}^{M_6 T})\tilde{\kappa}(T) \\ &= (1 + M_5 \mathrm{e}^{M_6 T}) \|\pi_2(\theta_T b)\|_{L(\mathcal{X},\mathcal{X})} M_8(1) C(K_1 + K_2) \mathrm{e}^{-\eta(T-1)} \, . \end{aligned}$$

Since $\eta > M_6$ and since π_2 is tempered from above in $\mathcal{X}$, there is a positive $T_1 \geq T_0$ with $\kappa(T) < 1$ for $T \geq T_1$. Thus, for $T \geq T_1$, G^T is a contractive self-mapping on the complete metric space $\mathcal{G}_L$.

The rest of the proof proceeds as in the first part of the proof.

3 Nonautonomous Evolution Equations

3.1 Two-Parameter Semi-Flow

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ be a Banach space. The solutions of a nonautonomous evolution equation will not generate a semi-flow but a two-parameter semi-flow.

Definition 21 (Two-parameter Semi-Flow). A *two-parameter semi-flow* μ on $\mathcal{X}$ is a continuous mapping

$$\{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathcal{X} : \; t \geq s\} \ni (t, s, x) \mapsto \mu(t, s, x) \in \mathcal{X}$$

which satisfies

(i) $\mu(s, s, \cdot) = \mathrm{id}_{\mathcal{X}}$ for $s \in \mathbb{R}$;

(ii) the two-parameter semi-flow property for $t \geq \tau \geq s$, $x \in \mathcal{X}$, i.e.

$$\mu(t, \tau, \mu(\tau, s, x)) = \mu(t, s, x) \, .$$

The next lemma explains how a two-parameter semi-flow defines an NDS.

Lemma 22 (Two-parameter Semi-Flow defines NDS). *Suppose that μ is a two-parameter semi-flow. Then $\varphi : \mathbb{R}_{\geq 0} \times \mathcal{B} \times \mathcal{X} \to \mathcal{X}$,*

$$\varphi(t, b)x = \mu(t + b, b, x) \tag{9}$$

is an NDS with base $\mathcal{B} = \mathbb{R}$ and driving system $\theta : \mathbb{R} \times \mathcal{B} \to \mathcal{B}$,

$$\theta(t)b = t + b \, .$$

Moreover, for $t \geq s$ and $x \in \mathcal{X}$ the relation $\mu(t, s, x) = \varphi(t - s, s)x$ holds.

Proof. θ is a dynamical system. We have

$$\varphi(0,b) = \mu(b,b,\cdot) = \mathrm{id}_{\mathcal{X}} .$$

We use the two-parameter semi-flow property of μ to obtain for $t, s \geq 0$, $b \in \mathcal{B}$

$$\begin{aligned}\varphi(t+s,b) &= \mu(t+s+b,b,\cdot)\\ &= \mu(t+s+b,s+b,\mu(s+b,b,\cdot))\\ &= \varphi(t,\theta_s b)\circ\varphi(s,b) ,\end{aligned}$$

proving the cocycle property of φ. The continuity of μ implies the continuity of $(t,x) \mapsto \varphi(t,b)x$. Now substitute t by $t-s$ and b by s in (9) to see that $\mu(t,s,x) = \varphi(t-s,s)x$.

Translating the definitions for nonautonomous dynamical systems to two-parameter semi-flows we obtain the following properties:

Condition 23 (Cone Invariance Property). There are $L > 0$ and $T_0 \geq 0$ such that for $\tau \in \mathbb{R}$ and $x, y \in \mathcal{X}$,

$$x - y \in \mathcal{C}_L(\tau) := \{\xi : \|\pi_2(\tau)\xi\|_{\mathcal{Y}} \leq L\|\pi_1(\tau)\xi\|_{\mathcal{Y}}\}$$

implies

$$\mu(t,\tau,x) - \mu(t,\tau,y) \in \mathcal{C}_L(t) \qquad \text{for } t \geq \tau + T_0 .$$

Condition 24 (Squeezing Property). There exist positive constants K_1, K_2 and η such that for every $\tau \in \mathbb{R}$, $x, y \in \mathcal{X}$ and $T > 0$ the identity

$$\pi_1(\tau+T)\mu(\tau+T,\tau,x) = \pi_1(\tau+T)\mu(\tau+T,\tau,y)$$

implies for all $x' \in \mathcal{X}$ with $\pi_1(\tau)x' = \pi_1(\tau)x$ and $x' - y \in \mathcal{C}_L(\tau)$ the estimates

$$\|\pi_i(t)\,[\mu(t,\tau,x) - \mu(t,\tau,y)]\,\|_{\mathcal{Y}} \leq K_i \mathrm{e}^{-\eta(t-\tau)}\|\pi_2(\tau)\,[x - x']\,\|_{\mathcal{Y}} , \quad i = 1,2,$$

for $t \in [\tau, \tau + T]$.

Condition 25 (Boundedness Property). For all $t, \tau \in \mathbb{R}$ with $t \geq \tau$ and all $M_1 \geq 0$ there exists a $M_2 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_2(\tau)x\|_{\mathcal{X}} \leq M_1$ the estimate

$$\|\pi_2(t)\mu(t,\tau,x)\|_{\mathcal{X}} \leq M_2$$

holds.

Condition 26 (Coercivity Property). For all $t, \tau \in \mathbb{R}$ with $t \geq \tau$ and all $M_3 \geq 0$ there exists a $M_4 \geq 0$ such that for $x \in \mathcal{X}$ with $\|\pi_1(\tau)x\|_{\mathcal{X}} \geq M_4$ the estimate

$$\|\pi_1(t)\mu(t,\tau,x)\|_{\mathcal{X}} \geq M_3$$

holds.

Condition 27 (Stationarity Property). There is a uniformly bounded invariant set $\mathcal{I}$.

Condition 28 (Strong Coercivity Property). For all $\tau \in \mathbb{R}$ there exist positive numbers M_5, M_6, M_7 such that for $x \in \mathcal{I}(\tau) + \mathcal{C}_L$ and all $t \geq \tau$ the estimate

$$\|\pi_1(\tau)x\|_{\mathcal{Y}} \leq M_5 \mathrm{e}^{M_6(t-\tau)}(M_7 + \|\pi_1(t)\mu(t,\tau,x)\|_{\mathcal{Y}})$$

holds.

Condition 29 (Smoothing Property). There are functions $M_8, M_9 :]0,\infty[\to]0,\infty[$ such that for $x, y \in \mathcal{X}$, $\tau \in \mathbb{R}$, and $t > \tau$ the Lipschitz estimates

$$\|\mu(t,\tau,x) - \mu(t,\tau,y)\|_{\mathcal{X}} \leq M_8(t-\tau)\|x-y\|_{\mathcal{Y}}$$

and

$$\|\pi_1(\tau)[x-y]\|_{\mathcal{X}} \leq M_9(t-\tau)\|\pi_1(t)[\mu(t,\tau,x) - \mu(t,\tau,y)]\|_{\mathcal{X}} \qquad \text{if } x-y \in \mathcal{C}_L \tag{10}$$

hold.

Theorem 30 (Inertial Manifold for Two-parameter Semi-Flow). *Suppose that μ is a two-parameter semi-flow on $\mathcal{X}$ and let $(\pi_i(\tau))_{\tau\in\mathbb{R}} \subset L(\mathcal{X})$, $i = 1, 2$, be two families of complementary projectors $\pi_1(\tau)$ and $\pi_2(\tau)$. Let μ satisfy the cone invariance and squeezing property. Moreover, let*

- *the boundedness and coercivity property*

or

- *the stationarity and strong coercivity property with respect to the invariant set $\mathcal{I}$ and the cone $\mathcal{C}_L$ with $M_6 < \eta$*

be satisfied.

If $\mathcal{X} \neq \mathcal{Y}$, we further assume that μ possesses the smoothing property, that π_1 is tempered from above in $\mathcal{X}$, and that there are constants M_{10} and M_{11} with

$$\|\pi_2(\tau)\|_{L(\mathcal{X},\mathcal{X})} \leq M_{10}\,, \quad \|\pi_1(\tau)x\|_{\mathcal{Y}} \leq M_{11}\|\pi_1(\tau)x\|_{\mathcal{X}} \qquad \text{for } x \in \mathcal{X},\ \tau \in \mathbb{R}\,.$$

Then there exists an inertial manifold $\mathcal{M} = (\mathcal{M}(\tau))_{\tau\in\mathbb{R}}$ of μ with the following properties:

(i) $\mathcal{M}(\tau)$ is a graph in $\pi_1(\tau)\mathcal{X} \oplus \pi_2(\tau)\mathcal{X}$,

$$\mathcal{M}(\tau) = \{x_1 + m(\tau, x_1) :\ x_1 \in \pi_2(\tau)\mathcal{X}\} \subset \mathcal{I}(\tau) + \mathcal{C}_L$$

with a mapping $m(\tau,\cdot) = m(\tau) : \pi_1(\tau)\mathcal{X} \to \pi_2(\tau)\mathcal{X}$ which is globally Lipschitz continuous

$$\|m(\tau,x_1) - m(\tau,y_1)\|_{\mathcal{X}} \leq \hat{L}\|x_1 - y_1\|_{\mathcal{X}}\,,$$

with some $\hat{L}$, and

$$\|m(\tau,x_1) - m(\tau,y_1)\|_{\mathcal{Y}} \leq L\|x_1 - y_1\|_{\mathcal{Y}}$$

with L from the cone invariance property.

(ii) $\mathcal{M}$ is exponentially attracting,

$$\|\mu(t,\tau,x)-\mu(t,\tau,x')\|_{\mathcal{X}} \le \hat{K}\mathrm{e}^{-\eta(t-\tau)}\|\pi_2(\tau)x-m(\tau,\pi_1(\tau)x)\|_{\mathcal{X}}$$

for $t \ge \tau+1$ with an asymptotic phase $x' = x'(\tau,x) \in \mathcal{M}(\tau)$ of x and some $\hat{K}$, moreover

$$\|\pi_i(t)[\mu(t,\tau,x)-\mu(t,\tau,x')]\|_{\mathcal{Y}} \le K_i\mathrm{e}^{-\eta(t-\tau)}\|\pi_2(\tau)x-m(\tau,\pi_1(\tau)x)\|_{\mathcal{Y}}$$

for $t \ge \tau$, $i=1,2$, and the constants $K_1, K_2 > 0$ from the squeezing property.

Proof. By Lemma 22, the two-parameter semi-flow μ defines an NDS φ with base $\mathcal{B}=\mathbb{R}$ and driving system $\theta : \mathbb{R}\times\mathcal{B}\to\mathcal{B}$ with $\theta(t)\tau = t+\tau$, $\tau = b \in \mathcal{B}$. Now Theorem 30 follows from Theorem 20.

In the next two subsections we verify the assumptions of Theorem 30 for evolution equations under the assumptions of exponential dichotomy conditions on the linear part or under the requirement that the linear part A is constant and selfadjoint such that we may use the eigenvalues of A.

3.2 Exponential Dichotomy Conditions

Let $\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$ be Banach spaces equipped with norms $\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$, $\|\cdot\|_{\mathcal{Z}}$, and let $(A(t))_{t\in\mathbb{R}}$ be a family of densely defined linear operators $A(t)$ on $\mathcal{Z}$ with domain $D(A(t))$ in $\mathcal{Z}$. We consider a nonautonomous evolution equation

$$\dot{x} + A(t)x = f(t,x) \tag{11}$$

which satisfies the following assumptions:

(A1) *Linearly $A(t)$:*

• Existence of evolution operator of the linear system: Under suitable additional assumptions on $\mathcal{X}$, $\mathcal{Z}$, A and f (see for example [Hen81], [DKM92], [Lun95]), we may define the evolution operator $\Phi : \{(t,s)\in\mathbb{R}^2 : t\ge s\} \to L(\mathcal{Z},\mathcal{Z})$ of the linear equation

$$\dot{x} + A(t)x = 0 \tag{12}$$

in $\mathcal{Z}$ as the solution of

$$\frac{d}{dt}\Phi(t,s) + A(t)\Phi(t,s) = 0 \qquad \text{for } t > s,\ s\in\mathbb{R}$$

and

$$\Phi(\tau,\tau) = \mathrm{id}_{\mathcal{Z}} \qquad \text{for } \tau\in\mathbb{R}\,.$$

• There are constants $k_0,\dots,k_4 \ge 1$, $\beta_2 > \beta_1$, $\gamma \in [0,1[$, a monotonously decreasing function $\psi \in C(\mathbb{R}_{>0},\mathbb{R}_{>0})$ with $\psi(t) \le k_0 t^{-\gamma}$, and a family $\pi_1 = (\pi_1(t))_{t\in\mathbb{R}}$ of linear, invariant projectors $\pi_1(t) : \mathcal{Z}\to\mathcal{Z}$, i.e.

$$\pi_1(t)\Phi(t,s) = \Phi(t,s)\pi_1(s) \qquad \text{for } t\ge s,$$

such that $\Phi(t,s)\pi_1(s)$ can be extended to a linear, bounded operator for $t \in \mathbb{R}$ with

$$\frac{d}{dt}\Phi(t,s)\pi_1(s) + A(t)\Phi(t,s)\pi_1(s) = 0 \qquad \text{for } t,s \in \mathbb{R}$$

and

$$\begin{aligned}
\|\Phi(t,s)\pi_1(s)\|_{L(\mathcal{Y},\mathcal{Y})} &\le k_1 \mathrm{e}^{-\beta_1(t-s)} && \text{for } t \le s\,,\\
\|\Phi(t,s)\pi_2(s)\|_{L(\mathcal{Y},\mathcal{Y})} &\le k_2 \mathrm{e}^{-\beta_2(t-s)} && \text{for } t \ge s\,,\\
\|\Phi(t,s)\pi_1(s)\|_{L(\mathcal{Z},\mathcal{Y})} &\le k_3 \mathrm{e}^{-\beta_1(t-s)} && \text{for } t \le s\,,\\
\|\Phi(t,s)\pi_2(s)\|_{L(\mathcal{Z},\mathcal{Y})} &\le k_4 \psi(t-s) \mathrm{e}^{-\beta_2(t-s)} && \text{for } t > s
\end{aligned} \tag{13}$$

with π_2, $\pi_2(t) = \mathrm{id}_{\mathcal{Z}} - \pi_1(t)$, as the complementary projector to π_1 in $\mathcal{Z}$.

For the case $\mathcal{X} \ne \mathcal{Y}$ we need the additional estimates

$$\begin{aligned}
\|\Phi(t,s)\pi_1(s)\|_{L(\mathcal{X},\mathcal{X})} &\le k_5 \mathrm{e}^{-\beta_1(t-s)} && \text{for } t \le s\,,\\
\|\Phi(t,s)\pi_2(s)\|_{L(\mathcal{X},\mathcal{X})} &\le k_5 \mathrm{e}^{-\beta_2(t-s)} && \text{for } t \ge s\,,\\
\|\Phi(t,s)\pi_1(s)\|_{L(\mathcal{Z},\mathcal{X})} &\le k_6 \mathrm{e}^{-\beta_1(t-s)} && \text{for } t \le s\,,\\
\|\Phi(t,s)\pi_2(s)\|_{L(\mathcal{Z},\mathcal{X})} &\le k_6 (t-s)^{-\alpha} \mathrm{e}^{-\beta_2(t-s)} && \text{for } t > s\,,\\
\|\Phi(t,s)\|_{L(\mathcal{Y},\mathcal{X})} &\le k_7 (t-s)^{-\gamma} \mathrm{e}^{-\beta_0(t-s)} && \text{for } t > s\,,\\
\|\Phi(t,s)\|_{L(\mathcal{Z},\mathcal{X})} &\le k_8 (t-s)^{-\alpha} \mathrm{e}^{-\beta_0(t-s)} && \text{for } t > s
\end{aligned} \tag{14}$$

and

$$\|\pi_2(\tau)\|_{L(\mathcal{X},\mathcal{X})} \le M_{10}\,, \quad \|\pi_1(\tau)x\|_{\mathcal{Y}} \le M_{11}\|\pi_1(\tau)x\|_{\mathcal{X}} \qquad \text{for } x \in \mathcal{X},\ \tau \in \mathbb{R}\,. \tag{15}$$

with nonnegative constants k_5, k_6, k_7, k_8, M_{10}, M_{11} and constants $\beta_0 > 0$, $\alpha \in [\gamma, 1[$.

(A2) *Nonlinearity $f(t,x)$:* The nonlinear function $f \in C(\mathbb{R} \times \mathcal{X}, \mathcal{Z})$ is assumed to satisfy the Lipschitz inequality

$$\|\pi_i(t)[f(t,x) - f(t,y)]\|_{\mathcal{Z}} \le \gamma_i(\|\pi_1(t)[x-y]\|_{\mathcal{Y}}, \|\pi_2(t)[x-y]\|_{\mathcal{Y}}) \tag{16}$$

for all $t \in \mathbb{R}$, $x, y \in \mathcal{X}$, where γ_i are suitable norms on $\mathbb{R}^2$.

(A3) *Existence of mild solutions:* We have the existence and uniqueness of the mild solutions

$$\mu(\cdot, \tau, \xi) \in C([\tau, \infty[, \mathcal{X})$$

of (11) with initial condition $x(\tau) = \xi \in \mathcal{X}$, i.e. let μ be the continuous solution of the integral equation

$$x(t) = \Phi(t,\tau)\xi + \int_\tau^t \Phi(t,r) f(\tau, x(r))\,\mathrm{d}r \qquad \text{for } t \ge \tau\,.$$

These were the assumptions.

Remark 31. Conditions like our assumptions can be found in the literature and they are standard for ordinary differential equations and for time-independent evolution equations in the non-selfadjoint case, see for example [Tem97]. For concrete examples of the realization of these assumptions we refer to Sec. 4, where we will apply our following Theorem 42 on the existence of inertial manifolds in these special situations.

Remark 32. 1. If $\mathcal{X} = \mathcal{Y}$ then we choose $k_i + 4 = k_i$ for $i = 1, 2, 3, 4$.

2. In the special cases $k_1 = k_2 = 1$, $\psi = 1$, $\gamma_i(w) = \ell_{i1}|w^1| + \ell_{i2}|w^2|$, the following considerations can be drastically simplified: For $k_1 = k_2 = 1$ we can show the cone invariance property with $T_0 = 0$ and constant $\tilde{L}$. If $\psi = 1$ we don't have a singularity in the integral inequalities for the estimation of solutions. If γ_1, γ_2 have the above mentioned structure then we can use linear comparison problems.

In order to apply Theorem 30, we have to show the cone invariance property and the squeezing property for the two-parameter semi-flow μ with respect to the projector π_1.

For fixed $r_1, r_2 \geq 0$ and $T \geq 0$, we define

$$(\Lambda^1 w)(t) := k_3 \int_t^T \mathrm{e}^{-\beta_1(t-r)} \gamma_1(w(r)) \,\mathrm{d}r \,,$$
$$(\Lambda^2 w)(t) := k_4 \int_0^t \psi(t-r) \mathrm{e}^{-\beta_2(t-r)} \gamma_2(w(r)) \,\mathrm{d}r + k_2 \mathrm{e}^{-\beta_2 t} L w^1(0)$$

and

$$q(t) := \left(k_1 \mathrm{e}^{-\beta_1(t-T)} r_1, k_2 \mathrm{e}^{-\beta_2 t} r_2 \right)$$

for $t \in [0, T]$ and $w \in C([0, T], \mathbb{R}^2)$. Then $q \in C([0, T], \mathbb{R}^2_{\geq 0})$. Because of $\psi(t) \leq k_0 t^{-\gamma}$ with $\gamma \in [0, 1[$, Λ is an at most weakly singular integral operator from $C([0, T], \mathbb{R}^2)$ into $C([0, T], \mathbb{R}^2)$. Moreover, Λ is completely continuous.

Lemma 33. *Assume there are $\tilde{L} :]0, \infty[\to]0, \infty[$, $L > 0$ and $T_0 \geq 0$ such that*

$$\tilde{L}(T) \leq L \qquad \text{for } T \geq T_0$$

and such that

$$v^2(T) \leq \tilde{L}(T) r_1 \tag{17}$$

holds for each solution $v \in C([0, T], \mathbb{R}^2_{\geq 0})$ of

$$v^i(t) \leq (\Lambda v)^i (t) + q^i(t) \qquad \text{for } i = 1, 2,\ t \in [0, T] \tag{18}$$

with $r_1 \geq 0$, $r_2 = 0$. Then μ possesses the cone invariance property with respect to π with the parameters $\tilde{L}$, L and T_0.

Proof. See [KS01] for $\mathcal{X} = \mathcal{Y}$.

Lemma 34. *Assume there are positive numbers L, η, K_1, K_2 such that*

$$v^i(t) \leq K_i \mathrm{e}^{-\eta t} r_2 \qquad \text{for } t \in [0, T] \tag{19}$$

holds for each $T > 0$ and each solution $v \in C([0, T], \mathbb{R}^2_{\geq 0})$ of (18) with $r_1 = 0$, $r_2 \geq 0$. Then μ possesses the squeezing property with respect to π with the parameters L, η, K_1, K_2.

Proof. See [KS01].

To estimate the solutions v of (18), we use the following comparison theorem for monotone iterations in ordered Banach spaces. The basic ideas and notions go back for example to [KLS89].

Let $\mathfrak{B}$ be a Banach space and let $\mathfrak{C}$ be an order cone in $\mathfrak{B}$. The order cone $\mathfrak{C}$ induces a semi-order $\leq_{\mathfrak{C}}$ in $\mathfrak{B}$ by

$$u \leq_{\mathfrak{C}} w \quad :\Longleftrightarrow \quad w - u \in \mathfrak{C} .$$

The norm in $\mathfrak{B}$ is called *semi-monotone* if there is a constant c with $\|x\|_{\mathfrak{B}} \leq c\|y\|_{\mathfrak{B}}$ for each $x, y \in \mathfrak{B}$ with $0 \leq_{\mathfrak{C}} x \leq_{\mathfrak{C}} y$. The cone $\mathfrak{C}$ is called normal if the norm in is semi-monotone, and $\mathfrak{C}$ is called solid if $\mathfrak{C}$ contains an open ball with positive radius.

Note that $C([0,T], \mathbb{R}^N_{\geq 0})$ is a normal, solid cone in $C([0,T], \mathbb{R}^N)$.

In a Banach space $\mathfrak{B}$ with normal and solid cone $\mathfrak{C}$, we study the fixed-point problem

$$u = Pu + p \tag{20}$$

with $p \in \mathfrak{B}$ and $P : \mathfrak{B} \to \mathfrak{B}$. We assume that P is completely continuous, increasing,

$$Pu \leq_{\mathfrak{C}} Pv \qquad \text{if } u \leq_{\mathfrak{C}} v ,$$

subadditive,

$$P(u+v) \leq_{\mathfrak{C}} Pu + Pv , \qquad u, v \in \mathfrak{C} ,$$

and homogeneous with respect to nonnegative factors,

$$P(\lambda u) = \lambda Pu \qquad \lambda \in \mathbb{R}_{\geq 0} , \; u \in \mathfrak{C} .$$

Definition 35. A function $w \in \mathfrak{B}$ is called *upper (lower) solution* of (20) if $Pw + p \leq_{\mathfrak{C}} w$ ($w \leq_{\mathfrak{C}} Pw + p$).

We need the existence of a unique solution $w \in \mathfrak{C}$ of (20) and an estimation of lower solutions $v \in \mathfrak{C}$ of (20) by solutions or upper solutions of (20).

Lemma 36. *Assume that there are $y \in \operatorname{int}\mathfrak{C}$ and $\delta \in [0, 1[$ with*

$$Py \leq_{\mathfrak{C}} \delta y .$$

Then there is a unique solution x_* of (20) in $\mathfrak{C}$ and

$$\underline{x} \leq_{\mathfrak{C}} x_* \leq_{\mathfrak{C}} \overline{x} \tag{21}$$

holds for each lower solution $\underline{x} \in \mathfrak{C}$ and each upper solution $\overline{x} \in \mathfrak{C}$ of (20).

Proof. See [KS01].

In order to apply Lemma 36 to our situation, we choose $\mathfrak{B} = C([0,T],\mathbb{R}^2)$ and $\mathfrak{C} = C([0,T],\mathbb{R}^2_{\geq 0})$. Then $\mathfrak{C}$ is a normal. The operator $P = \Lambda$ is increasing and completely continuous, and $p = q$ belongs to $\mathfrak{C}$. So we only have to find a function w^* in the interior of $\mathfrak{C}$ with

$$\Lambda w^* \leq_{\mathfrak{C}} \varepsilon w^* \qquad \text{with some } \varepsilon \in [0,1[\,. \tag{22}$$

Further we can estimate the solutions v of (18) by solutions $\bar{w} \in \mathfrak{C}$ of

$$\Lambda \bar{w} + q \leq_{\mathfrak{C}} \bar{w}\,. \tag{23}$$

Lemma 37. *Let $t_* \geq 0$ be fixed with*

$$\psi_* := \lim_{t \to t_*} \psi(t) < \infty\,, \quad \psi(t) > \psi_* \quad \textit{for } t < t_*\,. \tag{24}$$

Further let

$$k_9 \geq \delta \int_0^{t_*} \psi(r) \mathrm{e}^{-\delta r}\,\mathrm{d}r + \psi_* \lim_{t \to t_*} \mathrm{e}^{-\delta t} \quad \textit{for all } \delta \in \,]0, \beta_2 - \beta_1[\,. \tag{25}$$

Assume that there are positive numbers $\rho_1 < \rho_2$ with

$$G(\rho_1) = G(\rho_2) = 0\,, \quad G(\rho)\big|_{[\rho_1,\rho_2]} \neq 0 \tag{26}$$

and

$$k_1 k_2 \rho_1 < k_9^{-1} \psi_* \rho_2 \tag{27}$$

where $G : \mathbb{R}_{>0} \to \mathbb{R}$ is defined by

$$G(\rho) := \beta_2 - \beta_1 - k_3 \gamma_1(1,\rho) - k_4 k_9 \rho^{-1} \gamma_2(1,\rho)\,.$$

Then there are positive numbers $\eta_1 < \eta_2$ with

$$\eta_i = \beta_1 + k_3 \gamma_1(1,\rho_i) = \beta_2 - k_4 k_9 \rho_i^{-1} \gamma_2(1,\rho_i)\,,$$

and the cone invariance and squeezing property hold with

$$\eta := \eta_2\,, \quad L \in \,]k_1 \rho_1, k_2^{-1} k_9^{-1} \psi_* \rho_2[\,,$$
$$K_1 := \frac{k_2 k_9}{\rho_2 \psi_* - k_2 k_9 L}\,, \quad K_2 := \rho_2 K_1$$

and

$$\tilde{L}(t) = k_1 \frac{(\rho_2 - \tilde{\rho})\rho_1 \mathrm{e}^{-\eta_1 t} + (\tilde{\rho} - \rho_1)\rho_2 \mathrm{e}^{-\eta_2 t}}{(\rho_2 - \tilde{\rho})\mathrm{e}^{-\eta_1 t} + (\tilde{\rho} - \rho_1)\mathrm{e}^{-\eta_2 t}}\,.$$

with some $\tilde{\rho} \in \,]\max\{\rho_1, k_2 k_9 \psi_*^{-1} L\}, \rho_2[$.

Remark 38. Because of $\lim_{\rho\to 0} G(\rho) = \lim_{\rho\to\infty} G(\rho) = -\infty$, the existence of $\rho_* > 0$ with $G(\rho_*) > 0$ implies the existence of positive numbers $\rho_1 < \rho_2$ with (26) $G(\rho_1) = G(\rho_2) = 0$ and $\rho_1 < \rho_* < \rho_2$. Since $G(\rho_1) = 0$ and $\rho_1 < \rho_*$ imply

$$\rho_1 < \frac{k_4 k_9 \rho_*^{-1} \gamma_2(1,\rho_*)}{\beta_2 - \beta_1 - k_3\gamma_1(1,\rho_*)} \,,$$

the inequality (27) holds if

$$\beta_2 - \beta_1 > k_3\gamma_1(1,\rho_*) + \frac{k_1 k_2 k_4 k_9^2}{\psi_* \rho_*}\gamma_2(1,\rho_*) \,. \tag{28}$$

Since (28) implies $G(\rho_*) > 0$, condition (26) can be replaced by (28) for some $\rho_* > 0$.

Proof. (of Theorem 42) We show that the two-parameter semiflow μ generated by (11) satisfies the assumptions of Theorem 22. By Lemma 39 or Lemma 40 it remains to show that the cone invariance and squeezing property are satisfied.

Step 1: Determining of Solutions of (22) and (23)
In order to find a solution w^* of (22), first we look for $w \in \mathfrak{C}$ in the form

$$w(t) = \mathrm{e}^{-\eta t}(1,\rho) \tag{29}$$

with $\rho > 0$ and satisfying

$$w^1(t) \geq (\Lambda w)^1(t) + c_1 \mathrm{e}^{-\beta_1(t-T)} \tag{30}$$

and

$$w^2(t) \geq (\Lambda w)^2(t) + (c_2\rho - k_2 L)\mathrm{e}^{-\beta_2 t} \tag{31}$$

for $t \in [0,T]$ with suitable positive c_1 and c_2.
If we assume $\eta > \beta_1$ and

$$\eta \geq \beta_1 + k_3\gamma_1(1,\rho) \tag{32}$$

then, because of

$$\begin{aligned}\mathrm{e}^{\eta t}(\Lambda w)^1(t) &= k_3\gamma_1(1,\rho)\int_t^T \mathrm{e}^{(\eta-\beta_1)(t-r)}\,\mathrm{d}r \\ &= \frac{k_3\gamma_1(1,\rho)}{\eta-\beta_1}\left(1 - \mathrm{e}^{(\eta-\beta_1)(t-T)}\right),\end{aligned}$$

we may choose

$$c_1 = c_1(\rho,\eta) := \frac{k_3\gamma_1(1,\rho)}{\eta-\beta_1}\mathrm{e}^{-\eta T} \tag{33}$$

in order to satisfy (30). Remains to satisfy (31).

Inserting (29) in (31) and dividing by $\rho e^{-\eta t}$, we have to satisfy

$$1 \geq H(t, \rho, \eta) \tag{34}$$

for $t \in [0, T]$ where $H : [0, \infty[\times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is defined by

$$H(t, \rho, \eta) := \frac{\gamma_2(1, \rho)}{\rho} k_4 \int_0^t \psi(r) e^{-(\beta_2 - \eta)r} \, dr + c_2 e^{-(\beta_2 - \eta)t} .$$

We choose

$$c_2 = c_2(\rho, \eta) := \frac{\gamma_2(1, \rho)}{(\beta_2 - \eta)\rho} k_4 \psi_* .$$

Because of

$$D_1 H(t, \rho, \eta) = \left(-(\beta_2 - \eta)c_2 + \frac{\gamma_2(1, \rho)}{\rho} k_4 \psi(t) \right) e^{-(\beta_2 - \eta)t}$$

and because of the monotonicity of ψ, the function $H(\cdot, \rho, \eta)$ is maximized at t_*. Hence we have

$$\begin{aligned} H(t, \rho, \eta) &\leq H(t_*, \rho, \eta) \\ &= \frac{\gamma_2(1, \rho)}{\rho} k_4 \left(\int_0^{t_*} \psi(r) e^{-(\beta_2 - \eta)r} dr + (\beta_2 - \eta)^{-1} \psi_* e^{-(\beta_2 - \eta)t_*} \right) \end{aligned}$$

for all $t \geq 0$. Because of (25), inequality (34) is satisfied if

$$\frac{\gamma_2(1, \rho)}{\rho} k_4 k_9 \leq \beta_2 - \eta . \tag{35}$$

Combining (32) with (35), we find

$$\beta_1 + k_3 \gamma_1(1, \rho) \leq \eta \leq \beta_2 - k_4 k_9 \rho^{-1} \gamma_2(1, \rho) \tag{36}$$

as a sufficient condition for (30) and (31).

By assumption there are positive numbers $\rho_1 < \rho_2$ with (26) and (27). Let

$$\eta_i := \beta_1 + k_3 \gamma_1(1, \rho_i) = \beta_2 - k_4 k_9 \rho_i^{-1} \gamma_2(1, \rho_i) .$$

Then (η_1, ρ_1) and (η_2, ρ_2) solve (36).

Moreover, $\eta_2 > \eta_1$. To show this, we note that $\eta_2 \geq \eta_1$ by the monotonicity of γ_1. Assuming $\eta_1 = \eta_2$ we find $\gamma_1(1, \rho_1) = \gamma_1(1, \rho_2)$ and $\gamma_2(\rho_1^{-1}, 1) = \gamma_2(\rho_2^{-1}, 1)$. By the convexity of the γ_1- and γ_2-balls we had $\gamma_1(1, \rho) = \gamma_1(1, \rho_1)$ and $\gamma_2(\rho^{-1}, 1) = \gamma_2(\rho_1^{-1}, 1)$ for $\rho \in [\rho_1, \rho_2]$. This would imply the constance of G on $[\rho_1, \rho_2]$ in contradiction to (26).

Because of (27) we can choose

$$L \in \,]k_1 \rho_1, k_2^{-1} k_9^{-1} \psi_* \rho_2[\, . \tag{37}$$

Now we define $w_i \in \mathfrak{C}$, $i = 1, 2$, by

$$w_i(t) := \mathrm{e}^{-\eta_i t}(1, \rho_i) \,. \tag{38}$$

Then

$$w_i^1(t) \geq (\Lambda w_i)^1 (t) + \frac{k_3 \gamma_1(1,\rho)}{\eta - \beta_1} \mathrm{e}^{-\eta T} \mathrm{e}^{-\beta_1 (t-T)}$$

and

$$w_i^2(t) \geq (\Lambda w_i)^2 (t) + \left(\rho_i k_9^{-1} \psi_* - k_2 L\right) \mathrm{e}^{-\beta_2 t}$$

on $[0, T]$ because of

$$c_1(\rho_i, \eta_i) = \frac{k_3 \gamma_1(1, \rho_i)}{\eta_i - \beta_1} = 1 \,, \quad c_2(\rho_i, \eta_i) = \frac{\gamma_2(1, \rho_i) k_4 \psi_*}{(\beta_2 - \eta_i)\rho_i} = k_9^{-1} \psi_* \,.$$

Because of (37) we have

$$\rho_2 k_9^{-1} \psi_* > k_2 L \tag{39}$$

and inequality (22) holds for $w^* := w_2$.

Let now $C_1 \in [0, 1]$, $C_2 > 0$ satisfy

$$\begin{aligned} C_2 \left(C_1 \mathrm{e}^{-\eta_1 T} + (1 - C_1)\mathrm{e}^{-\eta_2 T}\right) &\geq k_1 r_1 \,, \\ C_2 \left(C_1 \rho_1 k_9^{-1} \psi_* + (1 - C_1)\rho_2 k_9^{-1} \psi_* - k_2 L\right) &\geq k_2 r_2 \,. \end{aligned} \tag{40}$$

Then

$$\bar{w} := C_2 \left(C_1 w_1 + (1 - C_1) w_2\right)$$

solves

$$\Lambda \bar{w} + q \leq_{\mathfrak{C}} \bar{w} \,,$$

and Lemma 36 implies

$$v \leq_{\mathfrak{C}} \bar{w}$$

for each solution $v \in \mathfrak{C}$ of (18).

Step 2: Verification of the Cone Invariance Property
Because of (39) we can fix

$$\tilde{\rho} \in \left]\max\{\rho_1, k_2 k_9 \psi_*^{-1} L\}, \rho_2\right[\,.$$

Let $r_2 = 0$ and $r_1 \geq 0$. Then (40) is satisfied with

$$C_1 := \frac{\rho_2 - \tilde{\rho}}{\rho_2 - \rho_1} \,, \quad C_2 := \frac{k_1 r_1}{C_1 \mathrm{e}^{-\eta_1 T} + (1 - C_1)\mathrm{e}^{-\eta_2 T}} \,.$$

Thus we find

$$v^2(t) \leq \bar{w}^2(t) = C_2 \left(C_1 w_1^2(t) + (1 - C_1) w_2^2(t)\right) = \bar{L}(\tilde{\rho}, t) r_1$$

for $t \in [0, T]$ with

$$\bar{L}(\tilde{\rho}, t) = k_1 \frac{(\rho_2 - \tilde{\rho})\rho_1 \mathrm{e}^{-\eta_1 t} + (\tilde{\rho} - \rho_1)\rho_2 \mathrm{e}^{-\eta_2 t}}{(\rho_2 - \tilde{\rho})\mathrm{e}^{-\eta_1 T} + (\tilde{\rho} - \rho_1)\mathrm{e}^{-\eta_2 T}} .$$

Especially we have

$$v^2(T) \leq \bar{L}(\tilde{\rho}, T) r_1 .$$

The inequalities $\eta_2 > \eta_1 > \beta_1$ imply

$$\bar{L}(\tilde{\rho}, T) \to k_1 \rho_1 \qquad \text{as } T \to \infty .$$

Hence there are $T_0 \geq 0$ and $L \geq 0$ with (17), if the additional inequality

$$k_1 \rho_1 < L$$

holds, which trivially follows from (37).

By Lemma 33, the cone invariance property of μ as required in Theorem 30 is verified with $\tilde{L}(t) := \overline{L}(\tilde{\rho}, t)$.

Step 3: Verification of the Squeezing Property

Now let $r_1 = 0$ and $r_2 \geq 0$. Then we may choose

$$C_1 := 0 , \quad C_2 := \frac{k_2 k_9 r_2}{\rho_2 \psi_* - k_2 k_9 L}$$

in order to satisfy (40). Thus we find

$$v(t) \leq \frac{k_2 r_2}{\rho_2 \psi_* - k_2 k_9 L} \mathrm{e}^{-\eta_2 t}(1, \rho_2) \qquad \text{for } t \in [0, T] .$$

Hence (19) holds with

$$\eta := \eta_2 , \quad K_1 := \frac{k_2 k_9}{\rho_2 \psi_* - k_2 k_9 L} , \quad K_2 := \rho_2 K_1$$

and L satisfying (37). By Lemma 34, the squeezing property of μ as required in Theorem 30 is verified.

Lemma 39. *Let f be globally bounded. Then two-parameter flow μ possesses the boundedness property and the coercivity property.*

Proof. First we verify the boundedness property: By the boundedness of f there is a number $F \geq 0$ with

$$\|f(x)\|_{\mathcal{Z}} \leq F \qquad \text{for } x \in \mathcal{X} .$$

Thus, for $\tau \in \mathbb{R}$, $t \geq \tau$, $x \in \mathcal{X}$,

$$\pi_2(t)\mu(t, \tau, x) = \Phi(t, \tau)\pi_2(\tau)x + \int_\tau^t \Phi(t, r)\pi_2(r) f(r, \mu(r, \tau, x))\,\mathrm{d}r$$

and, by the exponential dichotomy conditions (13),

$$
\begin{aligned}
\|\pi_2(t)\mu(t,\tau,x)\|_{\mathcal{X}} \le \|\Phi(t,\tau)\pi_2(\tau)\|_{L(\mathcal{X},\mathcal{X})}\|\pi_2(\tau)x\|_{\mathcal{X}} \\
+ \int_\tau^t \|\Phi(t,r)\pi_2(r)\|_{L(\mathcal{Z},\mathcal{X})}\|f(r,\mu(r,\tau,x))\|_{\mathcal{X}}\,\mathrm{d}r \\
&\le k_5 \mathrm{e}^{-\beta_2(t-\tau)}\|\pi_2(\tau)x\|_{\mathcal{X}} + Fk_6\int_\tau^t \psi(t-r)\mathrm{e}^{-\beta_2(t-r)}\,\mathrm{d}r \\
&= k_5 \mathrm{e}^{-\beta_2(t-\tau)}\|\pi_2(\tau)x\|_{\mathcal{X}} + Fk_6\int_0^{t-\tau} \psi(r)\mathrm{e}^{-\beta_2(r)}\,\mathrm{d}r \\
&\le k_5 \mathrm{e}^{-\beta_2(t-\tau)}\|\pi_2(\tau)x\|_{\mathcal{X}} + Fk_6\int_0^{\infty} \psi(r)\mathrm{e}^{-\beta_2(r)}\,\mathrm{d}r\,.
\end{aligned}
$$

Thus, for any t,τ with $t \ge \tau$ and any $M_1 \ge 0$ there is an $M_2 \ge 0$ such that for $x \in \mathcal{X}$ with $\|\pi_2(\tau)x\|_{\mathcal{X}} \le M_1$ we have $\|\pi_2(t)\mu(t,\tau,x)\|_{\mathcal{X}} \le M_2$, i.e. the two-parameter flow possesses the boundedness property of μ as required in Theorem 22.

Now we verify the coercivity property: For $\tau \in \mathbb{R}$, $t \in [\tau,\tau+T]$, $x \in \mathcal{X}$, we have

$$
\begin{aligned}
\pi_1(t)\mu(t,\tau,x) &= \Phi(t,\tau+T)\pi_1(\tau+T)\mu(\tau+T,\tau,x) \\
&\quad + \int_{\tau+T}^t \Phi(t,r)\pi_1(r)f(r,\mu(r,\tau,x))\,\mathrm{d}r
\end{aligned}
$$

and hence

$$
\begin{aligned}
\pi_1(\tau)x &= \Phi(\tau,\tau+T)\pi_1(\tau+T)\mu(\tau+T,\tau,x) \\
&\quad + \int_{\tau+T}^{\tau} \Phi(\tau,r)\pi_1(r)f(r,\mu(r,\tau,x))\,\mathrm{d}r\,.
\end{aligned}
$$

The exponential dichotomy conditions (13) imply

$$
\begin{aligned}
\|\pi_1(\tau)x\|_{\mathcal{X}} &\le \|\Phi(\tau,\tau+T)\pi_1(\tau+T)\|_{L(\mathcal{X},\mathcal{X})}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} \\
&\quad + F\int_\tau^{\tau+T} \|\Phi(\tau,r)\pi_1(r)\|_{L(\mathcal{Z},\mathcal{X})}\,\mathrm{d}r \\
&\le k_5\mathrm{e}^{\beta_1 T}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} + Fk_6\int_\tau^{\tau+T} \mathrm{e}^{-\beta_1(\tau-r)}\,\mathrm{d}r \\
&= k_5\mathrm{e}^{\beta_1 T}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} + \frac{Fk_6}{\beta_1}(\mathrm{e}^{\beta_1 T}-1)\,.
\end{aligned}
$$

Hence

$$
\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{X}} \ge \frac{1}{k_5}\|\pi_1(\tau)x\|_{\mathcal{X}} - \frac{Fk_6}{\beta_1 k_5}(1-\mathrm{e}^{-\beta_1 T})
$$

for $T \ge 0$, $\tau \in \mathbb{R}$, $x \in \mathcal{X}$ which shows the coercivity property of the two-parameter flow μ as required in Theorem 22.

Lemma 40. *Let the cone invariance property be satisfied with a function $\tilde{L}$ which is bounded by $\hat{L}$. Let $(\mathcal{I}(\tau))_{\tau\in\mathbb{R}}$ be an invariant set which is uniformly bounded in τ. Then the two-parameter flow μ possesses the strong coercivity property with respect to $\mathcal{C}_L$ and $\mathcal{I}$ with $M_6 < \eta_2$.*

Proof. By the uniform boundedness of $\mathcal{I}$ there is a number $M_7 > 0$ with

$$\|\pi_1(t)x\|_{\mathcal{Y}} \le M_7\,, \quad \|\pi_2(t)x\|_{\mathcal{Y}} \le M_7 \qquad \text{for all } t\in\mathbb{R}\,, \quad x\in\mathcal{I}(t)\,.$$

Let $\tau\in\mathbb{R}$ and $x\in\mathcal{I}(\tau)+\mathcal{C}_L$. The forward invariance of $\mathcal{I}$ and the cone invariance property imply

$$\|\pi_2(t)\mu(t,\tau,x)\|_{\mathcal{Y}} \le M_7 + \tilde{L}(t-\tau)\left(M_7 + \|\pi_1(t)\mu(t,\tau,x)\|_{\mathcal{Y}}\right) \qquad \text{for } t>\tau\,.$$

Let $x_1,x_2\in\mathcal{X}$ with $x_1-x_2\in\mathcal{C}_L$ and $x_2\in\mathcal{I}(\tau)$. Let $\mu_\Delta(t) := \mu(t,\tau,x_1) - \mu(t,\tau,x_2)$. Because of

$$\begin{aligned}\pi_1(t)\mu(t,\tau,x_i) &= \Phi(t,\tau+T)\pi_1(\tau+T)\mu(\tau+T,\tau,x_i)\\ &\qquad + \int_{\tau+T}^{t}\Phi(t,r)\pi_1(r)f(r,\mu(r,\tau,x_i))\,\mathrm{d}r\,,\end{aligned}$$

the exponential dichotomy conditions (13) imply

$$\begin{aligned}\|\pi_1(t)\mu_\Delta(t)\|_{\mathcal{Y}} &\le \|\Phi(t,\tau+T)\pi_1(\tau+T)\mu_\Delta(\tau+T)\|_{\mathcal{Y}}\\ &\qquad + \int_t^{\tau+T}\|\Phi(t,r)\pi_1(r)[f(r,\mu(r,\tau,x_1))-f(r,\mu(r,\tau,x_2))]\|_{\mathcal{Y}}\,\mathrm{d}r\\ &\le k_1\mathrm{e}^{-\beta_1(t-\tau-T)}\|\pi_1(\tau+T)\mu_\Delta(\tau+T)\|_{\mathcal{Y}}\\ &\qquad + k_3\int_t^{\tau+T}\mathrm{e}^{-\beta_1(t-r)}\gamma_1(\|\pi_1(r)\mu_\Delta(r)\|_{\mathcal{Y}},\|\pi_2(r)\mu_\Delta(r)\|_{\mathcal{Y}})\,\mathrm{d}r\\ &\le k_1\mathrm{e}^{-\beta_1(t-\tau-T)}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x)\|_{\mathcal{Y}}\\ &\qquad + k_3\int_t^{\tau+T}\mathrm{e}^{-\beta_1(t-r)}\gamma_1(1,\tilde{L}(r-\tau))\|\pi_1(r)\mu_\Delta(r)\|_{\mathcal{Y}}\,\mathrm{d}r\,.\end{aligned}$$

Setting

$$u(s) = \|\pi_1(\tau+T-s)\mu_\Delta(\tau+T-s)\|_{\mathcal{Y}}\mathrm{e}^{-\beta_1 s} \qquad \text{for } s\in[0,T]\,,$$

we find the Gronwall inequality

$$u(s) \le k_1u(0) + k_3\int_0^s\gamma_1(1,\tilde{L}(T-\sigma))u(\sigma)\,\mathrm{d}\sigma \qquad \text{for } s\in[0,T]\,.$$

Hence

$$\begin{aligned}u(s) &\le k_1u(0)\left(1+k_3\int_0^s\mathrm{e}^{k_3\int_\tau^s\gamma_1(1,\tilde{L}(T-\sigma))\,\mathrm{d}\sigma}\gamma_1(1,\tilde{L}(T-\tau))\,\mathrm{d}\tau\right)\\ &= k_1u(0)\mathrm{e}^{k_3\int_0^s\gamma_1(1,\tilde{L}(T-\sigma))\,\mathrm{d}\sigma}\end{aligned}$$

and therefore

$$\|\pi_1(\tau)\mu_\Delta(\tau)\|_{\mathcal{Y}} \le M_9(T)\|\pi_1(\tau+T)\mu_\Delta(\tau+T)\|_{\mathcal{Y}} \tag{41}$$

with

$$\tilde{M}_6(T) := \beta_1 T + k_3 \int_0^T \gamma_1(1, \tilde{L}(T-\sigma))\,\mathrm{d}\sigma\ , \quad M_9(T) := k_1 \mathrm{e}^{\tilde{M}_6(T)}$$

Let $\rho^* \in]k_1\rho_1, \rho_2[$ be a number with $\gamma_1(1,\rho^*) < \gamma_1(1,\rho_2)$. To show the existence of ρ^*, we note that $\eta_2 = \beta_1 + \gamma_1(1,\rho_2) > \eta_1 = \beta_1 + \gamma_1(1,\rho_1)$. Assuming $\gamma_1(1,\rho) = \gamma_1(1,\rho_2)$ for $\rho \in [k_1\rho_1, \rho_2]$, the convexity of the γ_1-balls would imply $\gamma_1(1,\rho) = \gamma_1(1,\rho_2)$ for $|\rho| \le \rho_2$ and hence $\gamma_1(1,\rho_1) = \gamma_1(1,\rho_2)$ in contradiction to $\eta_1 < \eta_2$. Therefore $\gamma_1(1,k_1\rho_1) < \gamma_1(1,\rho_2)$ and by the continuity of γ_1 the existence of k_1 follows.

Since $\tilde{L}$ is monotonously decreasing with $\tilde{L}(0) = k_1\tilde{\rho}$ and $\lim_{t\to\infty}\tilde{L}(t) = k_1\rho_1 < \rho^* < \rho_2$, there is a $T^* \ge 0$ with $\tilde{L}(t) \le \rho^*$ for $t \ge T^*$. Thus

$$\begin{aligned}
\frac{1}{T}\tilde{M}_6(T) &= \beta_1 + k_3\frac{1}{T}\int_0^T \gamma_1(1,\tilde{L}(T-\sigma))\,\mathrm{d}\sigma \\
&\le \beta_1 + k_3\left(\frac{T^*}{T}\gamma_1(1,k_1\tilde{\rho}) + \gamma_1(1,\rho^*)\right) \\
&= M_6 + \frac{1}{T}\tilde{M}_5
\end{aligned}$$

for $T > 0$ where

$$M_6 := \beta_1 + k_3\gamma_1(1,\rho^*) < \beta_1 + k_3\gamma_1(1,\rho_2) = \eta_2$$

and

$$\tilde{M}_5 := T^*\gamma_1(1,k_1\tilde{\rho})\ .$$

Hence

$$\|\pi_1(\tau)\mu_\Delta(\tau)\|_{\mathcal{Y}} \le k_1\mathrm{e}^{\tilde{M}_5 + M_6 T}\|\pi_1(\tau+T)\mu_\Delta(\tau+T)\|_{\mathcal{Y}}\ .$$

Thus

$$\begin{aligned}
&\|\pi_1(\tau)\mu(\tau,\tau,x_1)\|_{\mathcal{Y}} \\
&\quad\le \|\pi_1(\tau)\mu_\Delta(\tau)\|_{\mathcal{Y}} + \|\pi_1(\tau)\mu(\tau,\tau,x_2)\|_{\mathcal{Y}} \\
&\quad\le M_7 + k_1\mathrm{e}^{\tilde{M}_5+M_6T}\|\pi_1(\tau+T)\mu_\Delta(\tau+T)\|_{\mathcal{Y}} \\
&\quad\le M_7 + k_1\mathrm{e}^{\tilde{M}_5+M_6T}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x_1)\|_{\mathcal{Y}} \\
&\qquad\quad + k_1\mathrm{e}^{\tilde{M}_5+M_6T}\|\pi_1(\tau+T)\mu(\tau+T,\tau,x_2)\|_{\mathcal{Y}} \\
&\quad\le M_7 + k_1\mathrm{e}^{\tilde{M}_5+M_6T}(M_7 + \|\pi_1(\tau+T)\mu(\tau+T,\tau,x_1)\|_{\mathcal{Y}}) \\
&\quad\le M_5\mathrm{e}^{M_6T}(M_7 + \|\pi_1(\tau+T)\mu(\tau+T,\tau,x_1)\|_{\mathcal{Y}})
\end{aligned}$$

with

$$M_5 := (1+k_1)\mathrm{e}^{\tilde{M}_5}\ .$$

Lemma 41. *The two-parameter flow μ possesses the smoothing Lipschitz property.*

Proof. Let $x, y \in \mathcal{X}$, $\tau \in \mathbb{R}$ and let $\mu_\Delta(t) = \mu(t,\tau,x) - \mu(t,\tau,y)$. By assumption $f(t,\cdot)$ is global Lipschitz from $\mathcal{X}$ to $\mathcal{Z}$ with some constant $\hat{L}$ independent of t. The exponential dichotomy conditions imply the generalized Gronwall inequality

$$\begin{aligned}
\|\mu_\Delta(t)\|_{\mathcal{X}} &\le \|\Phi(t,\tau)[x-y]\|_{\mathcal{X}} + \int_\tau^t \|\Phi(t,s)[f(s,\mu(s,\tau,x)) - f(s,\mu(s,\tau,x))]\|_{\mathcal{X}}\,\mathrm{d}s \\
&\le \|\Phi(t,\tau)\|_{L(\mathcal{Y},\mathcal{X})}\|x-y\|_{\mathcal{Y}} + \hat{L}\int_\tau^t \|\Phi(t,s)\|_{L(\mathcal{Z},\mathcal{X})}\|\mu_\Delta(s)\|_{\mathcal{X}}\,\mathrm{d}s \\
&\le k_7(t-\tau)^{-\gamma}\mathrm{e}^{\beta_0(t-\tau)}\|x-y\|_{\mathcal{Y}} + \hat{L}k_8\int_\tau^t (t-s)^{-\alpha}\mathrm{e}^{\beta_0(t-s)}\|\mu_\Delta(s)\|_{\mathcal{X}}\,\mathrm{d}s
\end{aligned}$$

for $\|\mu_\Delta(\cdot)\|_{\mathcal{X}}$. As proved for example in [Hen81], there is a function $M_8 :]0,\infty[\to]0,\infty[$ with

$$\|\mu_\Delta(t)\|_{\mathcal{X}} \le M_8(t-\tau)\|x-y\|_{\mathcal{Y}} \qquad \text{for } t > \tau\,.$$

Inequality (10) is shown in the proof of the previous Lemma as inequality (41).

Now we are in a position to state the following theorem as a direct consequence of Theorem 22 and the previous lemmata:

Theorem 42. *Let the assumptions of Lemma 37 and the assumptions of Lemma 39 or Lemma 40 be satisfied. Then the claim of Theorem 22 holds for the two-parameter semi-flow μ generated by (11) with η, L, K_1 and K_2 as given in Lemma 37.*

3.3 Indefinite Quadratic Forms

Let $\mathcal{H} = \mathcal{Z}$ be a Hilbert space equipped with norm $|\cdot|$, and let A be a densely defined linear operator on $\mathcal{H}$ which is selfadjoint, positive and which has compact resolvent. We consider a nonautonomous parabolic evolution equation

$$\dot{x} + Ax = f(t,x) \tag{42}$$

where the nonlinear part $f : \mathbb{R} \times \mathcal{X} \to \mathcal{H}$ satisfies the following assumptions:

- The Hilbert space $\mathcal{X} = D(A^\alpha)$ with norm $\|u\|_{\mathcal{X}} := |u|_\alpha := |A^\alpha u|$ is the domain of a power A^α of A with some $\alpha \in [0,1[$.
- $f(t,x)$ is locally Hölder continuous in t and global Lipschitz continuous in x.

Then there are maximally defined (classical) solutions

$$\mu(\cdot,\tau,\xi) \in C([\tau,\infty[,\mathcal{X}) \cap C^1(]\tau,\infty[,\mathcal{H})$$

of (42) with initial condition $x(\tau) = \xi$, see [Hen81], [Mik98], and (42) generates a two-parameter semi-flow μ on $\mathcal{X}$.

Let $\lambda_1 \le \lambda_2 \le \cdots$ denote the eigenvalues of A counted with their multiplicity and let e_1, e_2, ... denote the corresponding eigenvectors of A.

We fix $N \in \mathbb{N}$. Let π_1 be the orthogonal projector from $\mathcal{H}$ onto $\operatorname{span}\{e_1, \dots, e_N\}$ and let $\pi_2 := \mathrm{id}_{\mathcal{H}} - \pi_1$.

For a fixed Banach space $\mathcal{Y}$ with $\mathcal{X} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{Z}$, we introduce the quadratic forms $Q_\rho : \mathcal{X} \to \mathbb{R}$

$$Q_\rho(x) = \|\pi_2 x\|_{\mathcal{Y}}^2 - \rho^2 \|\pi_1 x\|_{\mathcal{Y}}^2 \qquad \text{for } x \in \mathcal{X}\,, \quad \rho > 0.$$

Our goal is to use inequalities

$$Q_\rho(\mu(t,\tau,x) - \mu(t,\tau,y)) \le \mathrm{e}^{-2\Lambda(\rho)(t-\tau)} Q_\rho(x-y) \tag{43}$$

for $t \ge \tau$, $x, y \in \mathcal{X}$ and suitable $\rho > 0$ in order to show the cone invariance and squeezing property.

Lemma 43. *Under the general assumption given above, let there exist $\rho_1 < \rho_2$, a function $\Lambda\colon\ [\rho_1, \rho_2] \to \mathbb{R}$ and a number $L_0 \in \,]\rho_1, \rho_2]$ with*

$$\Lambda(L_0) > 0$$

and (43) for $\rho \in [\rho_1, \rho_2]$, $t \ge \tau$, $x, y \in \mathcal{X}$. Then the two-parameter semi-flow μ possesses the cone invariance property for all $L \in [\rho_1, \rho_2]$ and the squeezing property with the parameters

$$L = \rho_1\,, \quad \eta = \Lambda(L_0)\,, \quad K_1 = \frac{\rho_2 L_0}{\sqrt{\rho_2^2 - L_0^2}\sqrt{L_0^2 - \rho_1^2}}\,, \quad K_2 = \rho_1 K_1\,.$$

Proof. 1. Let $\rho \in [\rho_1, \rho_2]$ and $\|\pi_2[x-y]\|_{\mathcal{Y}} \le \rho \|\pi_1[x-y]\|_{\mathcal{Y}}$. Then $Q_\rho(x-y) \le 0$ and hence by assumption $Q_\rho(\mu(t,\tau,x) - \mu(t,\tau,y)) \le 0$, i.e.,

$$\|\pi_2[\mu(t,\tau,x) - \mu(t,\tau,y)]\|_{\mathcal{Y}} \le \rho \|\pi_1[\mu(t,\tau,x) - \mu(t,\tau,y)]\|_{\mathcal{Y}} \quad \text{for all } t \ge \tau\,, \tag{44}$$

i.e., the cone invariance property is satisfied in $\mathcal{Y}$ for any parameter $L = \rho \in [\rho_1, \rho_2]$.

2. Let $L = \rho_1$ and let $\tau \in \mathbb{R}$, $T \ge 0$, and $x, y, z \in \mathcal{X}$ with $\pi_1 \mu(\tau+T, \tau, x) = \pi_1 \mu(\tau+T, \tau, y)$ and $\|\pi_2[x-z]\|_{\mathcal{Y}} \le L\|\pi_1[x-y]\|_{\mathcal{Y}}$. Assuming $Q_\rho(\mu(t,\tau,x) - \mu(t,\tau,y)) < 0$ for some $\rho \in [\rho_1, \rho_2]$, $t \in [\tau, \tau+T]$, we get a contradiction to

$$Q_\rho(\mu(t,\tau,x) - \mu(t,\tau,y)) = \|\pi_2[\mu(t,\tau,x) - \mu(t,\tau,y)]\|_{\mathcal{Y}}^2 \ge 0\,.$$

Hence,

$$0 \le Q_\rho(\mu(t,\tau,x) - \mu(t,\tau,y)) \le \mathrm{e}^{-2\Lambda(\rho)(t-\tau)} Q_\rho(x-y) \quad \text{for all } t \in [\tau, \tau+T]\,. \tag{45}$$

Using this inequality and setting $\mu_\Delta(t) := \mu(t,\tau,x) - \mu(t,\tau,y)$, we find

$$\begin{aligned} Q_\rho(\mu_\Delta(t)) &= \|\pi_2 \mu_\Delta(t)\|_{\mathcal{Y}}^2 - \rho^2 \|\pi_1 \mu_\Delta(t)\|_{\mathcal{Y}}^2 \\ &= (1 - \rho^2 \rho_2^{-2}) \|\pi_2 \mu_\Delta(t)\|_{\mathcal{Y}}^2 + \rho^2 \rho_2^{-2} \left(\|\pi_2 \mu_\Delta(t)\|_{\mathcal{Y}}^2 - \rho_2^2 \|\pi_1 \mu_\Delta(t)\|_{\mathcal{Y}}^2 \right) \\ &\ge (1 - \rho^2 \rho_2^{-2}) \|\pi_2 \mu_\Delta(t)\|_{\mathcal{Y}}^2\,, \end{aligned}$$

i.e.,

$$\|\pi_2[\mu(t,\tau,x)-\mu(t,\tau,y)]\|_{\mathcal{Y}}^2 \le \frac{\rho_2^2}{\rho_2^2-\rho^2}\mathrm{e}^{-2\Lambda(\rho)t}Q_\rho(x-y)\rho\in[\rho_1,\rho_2[,\ t\in[\tau,\tau+T]\,.$$

for all $\rho\in[\rho_1,\rho_2[$, $t\in[\tau,\tau+T]$. With the first inequality in (45) and

$$\begin{aligned} Q_\rho(x-y') &= \|\pi_2[x-y]\|_{\mathcal{Y}}^2-\rho^2\|\pi_1[x-y]\|_{\mathcal{Y}}^2 \\ &\le \left(\|\pi_2[y-z]\|_{\mathcal{Y}}+\|\pi_2[x-z]\|_{\mathcal{Y}}\right)^2-\rho^2\|\pi_1[x-y]\|_{\mathcal{Y}}^2 \\ &\le \left(\|\pi_2[y-z]\|_{\mathcal{Y}}+L\|\pi_1[x-y]\|\right)^2-\rho^2\|\pi_1[x-y]\|_{\mathcal{Y}}^2 \\ &\le (1+\varepsilon^{-1})\|\pi_2[y-z]\|_{\mathcal{Y}}^2+\left((1+\varepsilon)L^2-\rho^2\right)\|\pi_1[x-y]\|_{\mathcal{Y}}^2 \\ &= \frac{\rho^2}{\rho^2-L^2}\|\pi_2[y-z]\|_{\mathcal{Y}}^2 \end{aligned}$$

for $\varepsilon=\rho^2L^{-2}-1$. Thus, for $\rho=L_0$, we get

$$\|\pi_i[\mu(t,\tau,x)-\mu(t,\tau,y)]\|_{\mathcal{Y}}\le K_i\mathrm{e}^{-t\eta}\|\pi_2[y-z]\|_{\mathcal{Y}}\quad \text{for all } t\in[\tau,\tau+T]\,,$$

i.e., the modified squeezing property is satisfied.

Theorem 44. *Let the assumptions of Lemma 43 be satisfied. Moreover, we assume that*

- *f is globally bounded*

or

- *there is an bounded invariant set $\mathcal{I}$.*

Then the claim of Theorem 22 holds for the two-parameter semi-flow μ generated by (11) with η, L, K_1 and K_2 as given in Lemma 43.

Proof. We only have to note that the exponential dichotomy conditions (13), (14) and the inequalities (15) can be satisfied for $\Phi(t,\tau)=\mathrm{e}^{-A(t-\tau)}$.

Remark 45. An inequality of the form (43) is used in [Rom94] for the special case $\mathcal{Y}=D(A^{\alpha/2})$. Assuming the Lipschitz inequality

$$|f(x)-f(y)|\le \ell|x-y|_\alpha \qquad x,y\in\mathcal{X}=D(A^\alpha) \tag{46}$$

and the spectral gap condition

$$\lambda_{N+1}-\lambda_N>\ell(\lambda_N^\alpha+\lambda_{N+1}^\alpha)\,, \tag{47}$$

A.V. Romanov [Rom94] shows that (43) holds for $\rho\in[h,h^{-1}]$, $\Lambda(\rho)=\lambda_{N+1}-\ell\lambda_{N+1}^\alpha$ and with $h<1$ satisfying

$$\lambda_{N+1}-\lambda_N>\ell(\lambda_N^\alpha+\frac{1}{2}(h^2+h^{-2})\lambda_{N+1}^\alpha)\,.$$

Thus our Theorem 44 allows to ensure the existence of inertial manifolds under the sharp spectral gap condition (47) for the Lipschitz inequality (46).

In the following two Lemmata we verify the assumption of Lemma 43. For simplicity we restrict us to $\mathcal{Y} = \mathcal{H}$ and to Lipschitz inequalities of the type

$$|f(t,x) - f(t,y)| \le \nu(x-y) \tag{48}$$

for all $x, y \in D(A)$, $t \in \mathbb{R}$ where

$$\nu(x) = \left(\sum_{i=1}^{M} d_i |x|_{\delta_i}^2\right)^{\frac{1}{2}}$$

with positive d_i, $i = 1, \ldots, M$, and

$$\gamma = \delta_1 \in [0, \min\{\alpha, \frac{1}{2}\}]\,, 0 \le \delta_{i+1} < \delta_i \qquad \text{for } i = 1, \ldots, M-1\,.$$

Let

$$g_1 := \left(\sum_{i=1}^{M} d_i \lambda_N^{2\delta_i}\right)^{\frac{1}{2}}\,, \quad g_2 := \left(\sum_{i=1}^{M} d_i \lambda_{N+1}^{2\delta_i}\right)^{\frac{1}{2}}$$

We show that in this case the assumptions of Lemma 43 and hence of Theorem 44 can be satisfied if the spectral gap condition

$$\lambda_{N+1} - \lambda_N > g_1 + g_2 \tag{49}$$

holds.

Especially, for

$$|f(t,x) - f(t,y)| \le \ell \cdot |x-y|_\gamma \qquad \text{for } x, y \in D(A)\,, t \in \mathbb{R}$$

we have the spectral gap condition

$$\lambda_{N+1} - \lambda_N > \ell(\lambda_{N+1}^\gamma + \lambda_N^\gamma)\,.$$

Let the auxiliary function $p : \mathbb{R} \to \mathbb{R}$ defined by

$$p(\rho) := (\lambda_{N+1} - \lambda_N)^2 \rho^2 - \left(\rho^2 + 1\right)\left(g_1^2 + \rho^2 g_2^2\right) \qquad \text{for } \rho \in \mathbb{R}\,. \tag{50}$$

Further, let

$$L_0 := \sqrt{g_1/g_2}\,.$$

Lemma 46. *Let the spectral gap condition (49) be satisfied. Then there are uniquely determined numbers $0 < \rho_1 < \rho_2$ with*

$$p(\rho_1) = p(\rho_2) = 0\,, \quad \rho_1 < L_0 < \rho_2\,.$$

The function $\Lambda :]0, \infty[\to \mathbb{R}$ defined by

$$\Lambda(\rho) := \lambda_{N+1} - \frac{\rho^2(1+\rho^2)g_2^2 + g_1^2 + \rho^2 g_2^2}{2L\sqrt{1+\rho^2}\sqrt{g_1^2 + \rho^2 g_2^2}} \qquad \textit{for } \rho > 0 \tag{51}$$

is maximized at L_0 with

$$\Lambda(L_0) = \lambda_{N+1} - g_2 > 0\,. \tag{52}$$

Proof. Because of

$$p(L_0) = \frac{g_1}{g_2}\left((\lambda_{N+1} - \lambda_N)^2 - (g_1 + g_2)^2\right) ,$$

the spectral gap condition (49) implies $p(L_0) > 0$. Since p is a quadratic polynomial in ρ^2 and $p(0) < 0$, the existence and uniqueness of zeroes ρ_1, ρ_2 of p in $]0, \infty[$ follows. Thus $p(\rho) > 0$ for $\rho \in]\rho_1, \rho_2[$ and $L_0 \in]\rho_1, \rho_2[$.

We have

$$\Lambda(\rho) = \lambda_{N+1} - \frac{1}{2} g_2 \left(H(\rho) + H(\rho)^{-1}\right)$$

with

$$H(\rho) := g_2 \left(\frac{1 + \rho^2}{\rho^{-2} g_1^2 + g_2^2}\right)^{\frac{1}{2}} ,$$

such that Λ has a global maximum on $]0, \infty[$ at ρ with $H(\rho) = 1$, i.e., at $\rho = L_0$. Since $\lambda_N > 0$, the spectral gap condition (49), we have (52).

Lemma 47. *Let $\mathcal{Y} = \mathcal{H}$ and* let (48) *and* (49) *be satisfied. Then*

$$Q_\rho(\mu(t, \tau, x) - \mu(t, \tau, y)) \leq Q_\rho(x - y) \mathrm{e}^{-2\Lambda(\rho)(t-\tau)}$$

for all $x, y \in \mathcal{X}$, $\rho \in [\rho_1, \rho_2]$ and $\tau \leq t$.

Proof. Let $\rho \in [\rho_1, \rho_2]$, $\tau < t$, $x, y \in \mathcal{X}$ be fixed. For shortness let $\mu_\Delta := \mu(t, \tau, x) - \mu(t, \tau, y)$, $f_\Delta := f(t, \mu(t, \tau, x) - f(t, \mu(t, \tau, y))$.

We have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} Q_\rho(\mu_\Delta) &= \Re \left\langle -A\mu_\Delta + f_\Delta, \pi_2\mu_\Delta - \rho^2 \pi_1 \mu_\Delta \right\rangle \\
&= -\langle A\mu_\Delta, \pi_2\mu_\Delta\rangle + \rho^2 \langle A\mu_\Delta, \pi_1\mu_\Delta\rangle + \Re \left\langle f_\Delta, \pi_2\mu_\Delta - \rho^2\pi_1\mu_\Delta \right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\Re \left\langle f_\Delta, \pi_2\mu_\Delta - \rho^2 \pi_1\mu_\Delta \right\rangle \\
&\quad \leq \nu(\mu_\Delta) |\pi_2\mu_\Delta - \rho^2\pi_1\mu_\Delta| \\
&\quad \leq \tfrac{\varepsilon}{2}\nu(\mu_\Delta)^2 + \tfrac{1}{2\varepsilon}|\pi_2\mu_\Delta - \rho^2\pi_1\mu_\Delta|^2 \\
&\quad \leq \tfrac{\varepsilon}{2}\nu(\pi_1\mu_\Delta)^2 + \tfrac{\varepsilon}{2}\nu(\pi_2\mu_\Delta)^2 + \tfrac{1}{2\varepsilon}|\pi_2\mu_\Delta|^2 + \tfrac{1}{2\varepsilon}\rho^4|\pi_1\mu_\Delta|^2 .
\end{aligned}$$

Note that

$$\langle A\mu_\Delta, \pi_1\mu_\Delta\rangle \leq \lambda_N |\pi_1\mu_\Delta|^2 , \quad -\langle A\mu_\Delta, \pi_2\mu_\Delta\rangle \leq -\lambda_{N+1}^{1-2\gamma} |\pi_2\mu_\Delta|_\gamma^2$$

and

$$\nu(\pi_1\mu_\Delta) \leq g_1|\pi_1\mu_\Delta| , \quad \nu(\pi_2\mu_\Delta) \leq g_2\lambda_{N+1}^{-\gamma}|\pi_2\mu_\Delta|_\gamma .$$

With $\epsilon > 0$, we estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Q_\rho(\mu_\Delta) &\le -\lambda_{N+1}^{1-2\gamma}|\pi_2\mu_\Delta|_\gamma^2 + \rho^2\lambda_N|\pi_1\mu_\Delta|^2 \\ &\qquad + \frac{\varepsilon}{2}\nu(\pi_1\mu_\Delta)^2 + \frac{\varepsilon}{2}\nu(\pi_2\mu_\Delta)^2 + \frac{1}{2\varepsilon}|\pi_2\mu_\Delta|^2 + \frac{1}{2\varepsilon}\rho^4|\pi_1\mu_\Delta|^2 \\ &\le |\pi_1\mu_\Delta|^2\left(\rho^2\lambda_N + \frac{1}{2\varepsilon}\rho^4 + \frac{\varepsilon}{2}g_1^2\right) \\ &\qquad \frac{1}{2\varepsilon}|\pi_2\mu_\Delta|^2 + |\pi_2\mu_\Delta|_\gamma^2\left(-\lambda_{N+1}^{1-2\gamma} + \frac{\varepsilon}{2}g_2^2\lambda_{N+1}^{-2\gamma}\right).\end{aligned}$$

Under the assumption

$$\lambda_{N+1} > \frac{\varepsilon}{2}g_2^2\,, \tag{53}$$

we find

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}Q_\rho(\mu_\Delta) &\le -\Lambda Q_\rho(\mu_\Delta) + |\pi_1\mu_\Delta|^2\left(-\Lambda\rho^2 + \rho^2\lambda_N + \frac{1}{2\varepsilon}\rho^4 + \frac{\varepsilon}{2}g_1^2\right) \\ &\qquad + |\pi_2\mu_\Delta|_\gamma^2\left(\Lambda + \frac{1}{2\varepsilon} - \lambda_{N+1} + \frac{\varepsilon}{2}g_2^2\right) \\ &\le -\Lambda Q_\rho(\mu_\Delta)\end{aligned} \tag{54}$$

if Λ satisfies

$$\lambda_N + \frac{1}{2\varepsilon}\rho^2 + \frac{\varepsilon}{2}g_1^2\rho^{-2} \le \Lambda \le \lambda_{N+1} - \frac{1}{2\varepsilon} - \frac{\varepsilon}{2}g_2^2\,.$$

This inequality is solvable with respect to Λ if and only if

$$\lambda_{N+1} - \lambda_N \ge \frac{1}{2\varepsilon}(1+\rho^2) + \frac{\varepsilon}{2}(g_1^2\rho^{-2} + g_2^2)\,. \tag{55}$$

The right-hand side is minimized at

$$\epsilon := \left(\frac{1+\rho^2}{\rho^{-2}g_1^2 + g_2^2}\right)^{\frac{1}{2}} \tag{56}$$

with the value

$$(1+\rho^2)^{\frac{1}{2}}(g_1^2\rho^{-2} + g_2^2)^{\frac{1}{2}}$$

such that we obtain the sufficient and necessary condition

$$\lambda_{N+1} - \lambda_N \ge (1+\rho^2)^{\frac{1}{2}}(g_1^2\rho^{-2} + g_2^2)^{\frac{1}{2}}\,. \tag{57}$$

By definition of p and by Lemma 46, this inequality holds for $\rho \in [\rho_1, \rho_2]$. So

$$\begin{aligned}\Lambda &= \lambda_{N+1} - \frac{1}{2\varepsilon} - \frac{\varepsilon}{2}g_2^2 \\ &= \lambda_{N+1} - \frac{1}{2}\left(\frac{g_1^2\rho^{-2} + g_2^2}{1+\rho^2}\right)^{\frac{1}{2}} - \frac{1}{2}\left(\frac{1+\rho^2}{g_1^2\rho^{-2} + g_2^2}\right)^{\frac{1}{2}} g_2^2 \\ &= \Lambda(\rho)\end{aligned}$$

solves (55) with ϵ given by (56).

Remains to show (53) with (56), i.e., we have to show

$$\sqrt{g_1^2+\rho^2 g_2^2}\lambda_{N+1} > \frac{1}{2}\rho\sqrt{1+\rho^2}g_2^2\,. \tag{58}$$

Indeed, (58) holds, since the inequalities (57) and $\lambda_{N+1} > \lambda_N > 0$ imply

$$\begin{aligned}\rho\sqrt{1+\rho^2}g_2^2 &\leq (\lambda_{N+1}-\lambda_N)\rho^2 g_2^2(g_1^2+\rho^2 g_2^2)^{-\frac{1}{2}}\\ &\leq \lambda_{N+1}(g_1^2+\rho^2 g_2^2)(g_1^2+\rho^2 g_2^2)^{-\frac{1}{2}}\\ &= \lambda_{N+1}\sqrt{g_1^2+\rho^2 g_2^2}\,.\end{aligned}$$

Summarizing we have that $\Lambda = \Lambda(\rho)$ satisfies (54). Therefore,

$$\frac{d}{dt}Q_\rho(\mu(t,\tau,x)-\mu(t,\tau,y)) \leq -2\Lambda(\rho)Q_\rho(x-y)$$

for all $t > \tau$, $x, y \in \mathcal{X}$, $\rho \in [\rho_1, \rho_2]$ such that the claim of the lemma follows.

Corollary 48. *Under the general assumptions of this section let f satisfy the Lipschitz inequality (48) with some $\gamma \in [0, \min\{\alpha, \frac{1}{2}\}]$ such that the spectral gap condition (49) holds. Moreover, we assume that*

- *f is globally bounded*

or

- *there is an bounded invariant set $\mathcal{I}$.*

Then the claim of Theorem 22 holds for the two-parameter semi-flow μ generated by (11) with η, L, K_1 and K_2 as given in Lemma 43.

4 Conclusion

Exponential dichotomy conditions of the form (13) are used, for example, in [Hen81], [Tem97], [BdMCR98], [LL99], [CS01]. There $k_3 = \beta_1^\alpha k_1$, $k_4 = \beta_2^\alpha$ with some $\alpha \in [0,1[$ depending on the spaces $\mathcal{X}$ and $\mathcal{Z}$, and $\psi(t) = \beta_2^{-\alpha}\max\{t^{-\alpha}, 1\}$, $\psi(t) = \beta_2^{-\alpha}t^{-\alpha}+1$, or $\psi(t) = \max\{\alpha^\alpha\beta_2^{-\alpha}t^{-\alpha}, 1\}$ where $0^0 := 1$. If A is a time-independent sectorial operator, then usually $\mathcal{X}$ is the domain $D((A+a)^\alpha)$ of the power $(A+a)^\alpha$ of $A+a$ with some $\alpha \in [0,1[$ and some $a \in \mathbb{R}$. If $\mathcal{X} = \mathcal{Z}$ then we may choose $\alpha = 0$ and $\psi = 1$.

In the special case that A is a time-independent, selfadjoint positive linear operator with compact resolvent and dense domain $D(A)$ on the Hilbert space $\mathcal{Z}$, usually one uses $\mathcal{X} = D(A^\alpha)$ with some $\alpha \in [0,1[$. Let π_1 be the orthogonal projector from $\mathcal{Z}$ onto the linear subspace spanned by the N eigenvectors of A corresponding to the first N eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ (counted with their multiplicity). Then we may choose $\beta_1 = \lambda_N$, $\beta_2 = \lambda_{N+1}$, $k_1 = k_2 = 1$, $k_3 = \beta_1^\alpha$, $k_4 = \beta_2^\alpha$, $\psi(t) := \max\{\alpha^\alpha\beta_2^{-\alpha}t^{-\alpha}, 1\}$, see for example [FST88, Lemma 3.1].

In [LL99] (and with $\mathcal{X} = \mathcal{Z}$), a Lipschitz inequality of the form

$$\|\pi_i(t)[f(t,x) - f(t,y)]\|_{\mathcal{X}} \le \ell_i \max\{\|\pi_1(t)[x-y]\|_{\mathcal{X}}, \|\pi_2(t)[x-y]\|_{\mathcal{X}}\}$$

is utilized. This special form of a Lipschitz inequality is contained in our Lipschitz assumption with $\gamma_i(w) = \ell_i|w|_\infty$ and $|\cdot|_\infty$ as the maximum norm in $\mathbb{R}^2$. The standard Lipschitz inequality

$$\|f(t,x) - f(t,y)\|_{\mathcal{Z}} \le \ell\|x-y\|_{\mathcal{X}}$$

in a Hilbert space $\mathcal{Z}$ and with orthogonal projectors $\pi_1(t)$ leads to (16) with $\gamma_i(w) = \ell|w|_2$ or $\gamma_i(w) = \ell|w|_1$, where $|\cdot|_1$ denotes the sum norm and $|\cdot|_2$ denotes the euclidean norm in $\mathbb{R}^2$.

In order to compare known results with ours we verify the assumptions of Theorem 42 for different forms of Lipschitz estimates for f and for concrete functions ψ in the exponential dichotomy property.

Corollary 49. *Under the general assumptions in Sec. 3.2, let f satisfy* (16) *with weighted maximum norms*

$$\gamma_i(w) = \ell_i \max\{|w^1|, |w^2|\}\,, \quad \ell_i > 0 \qquad \textit{for } w \in \mathbb{R}^2\,. \tag{59}$$

Let t_*, ψ_* and k_9 with the properties as in Theorem 42. Then condition (28) *and hence the claim of Theorem 42 hold if*

$$\beta_2 - \beta_1 > \frac{k_3\ell_1 + k_4k_9\ell_2}{2} + \sqrt{\frac{(k_3\ell_1 - k_4k_9\ell_2)^2}{4} + \frac{k_1k_2k_3k_4k_9^2\ell_1\ell_2}{\psi_*}}\,. \tag{60}$$

Proof. Calculating the zeroes of G with $G(\rho) = \beta_2 - \beta_1 - k_3\ell_1\max\{1,\rho\} - k_4k_9\ell_2\rho^{-1}\max\{1,\rho\}$, we find (60) as sufficient and necessary condition for (26), (27).

Latushkin and Layton [LL99] consider $-A$ as generator of a strongly continuous semigroup on the Banach space $\mathcal{X} = \mathcal{Z}$. Let $\mathcal{X}$ be the direct sum of two subspace $\mathcal{X}_1$ and $\mathcal{X}_2$ and let π_i the projector from $\mathcal{X}$ onto $\mathcal{X}_i$. Assuming exponential dichotomy conditions (13) with $k_1 = k_2 = 1$ (and $k_3 = k_4 = 1$, $\psi = 1$ because of $\mathcal{X} = \mathcal{Z}$) and $f(0) = 0$ and

$$\|\pi_i[f(x) - f(y)]\|_{\mathcal{X}} \le \ell_i \max\{\|\pi_1[x-y]\|_{\mathcal{X}}, \|\pi_2[x-y]\|_{\mathcal{X}}\}$$

for the time-independent nonlinearity f, they found

$$\beta_2 - \beta_1 > \ell_1 + \ell_2 \tag{61}$$

as optimal spectral gap condition. They extended this result to $(-A(t))$ as a family of linear operators on the Banach space $\mathcal{X} = \mathcal{Z}$ generating a strongly continuous semiflow, see [LL99] too. Again, assuming exponential dichotomy conditions (13) with $k_1 = k_2 = k_3 = k_4 = 1$, $\psi = 1$, and the Lipschitz estimate

$$\|\pi_i(t)[f(t,x) - f(t,y)]\|_{\mathcal{X}} \leq \ell_i \max\{\|\pi_1(t)[x-y]\|_{\mathcal{X}}, \|\pi_2(t)[x-y]\|_{\mathcal{X}}\}$$

and $f(t,0) = 0$ for f, they found the spectral gap condition (61) for nonautonomous inertial manifolds.

Since $\mathcal{X} = \mathcal{Z}$, we have $k_3 = k_1$, $k_4 = k_2$, $\psi = 1$. Thus we have to choose $t_* = 0$ and find $k_9 = \psi_* = 1$. Our condition (60) reduces to

$$\beta_2 - \beta_1 > k_1\ell_1 + k_2\ell_2\,,$$

which in the special case of $k_1 = k_2 = 1$ reduces to the optimal spectral gap condition (61) found by Y. Latushkin and B. Layton, [LL99].

Corollary 50. *Under the general assumptions in Sec. 3.2, let f satisfy (16) with weighted sum norms*

$$\gamma_i(w) = \ell_{i1}|w^1| + \ell_{i2}|w^2|\,, \quad \ell_{i1}, \ell_{i2} > 0 \qquad \text{for } w \in \mathbb{R}^2. \tag{62}$$

Let t_*, ψ_* and k_9 with the properties as in Theorem 42. Then condition (28) and hence the claim of Theorem 42 hold if

$$\beta_2 - \beta_1 > k_3\ell_{11} + k_4k_9\ell_{22} + \frac{k_1k_2k_9 + \psi_*}{\sqrt{k_1k_2\psi_*}}\sqrt{\ell_{12}\ell_{21}k_3k_4}\,. \tag{63}$$

Proof. Calculating the zeroes of G with $G(\rho) = \beta_2 - \beta_1 - k_3\ell_{11} - k_3\ell_{12}\rho - k_4k_9\ell_{21}\rho^{-1} - k_4k_9\ell_{22}$, we find (63) as a sufficient and necessary condition for (26), (27).

First let

$$\psi(t) := \max\{\alpha^\alpha\beta_2^{-\alpha}t^{-\alpha}, 1\}$$

as in [FST88, Lemma 3.1]. Here and in the following we set $0^0 := 1$ in order to continuously extend the expression for ψ to the limit case $\alpha = 0$. We choose $t_* := \alpha\beta_2^{-1}$ and hence we have $\psi_* = 1$. To satisfy (25) we note that

$$\delta\int_0^{t_*}\psi(r)\mathrm{e}^{-\delta r}\,\mathrm{d}r + \psi_*\lim_{t\to t_*}\mathrm{e}^{-\delta t} = \delta^\alpha\alpha^\alpha\beta_2^{-\alpha}\int_0^{\delta\alpha\beta_2^{-1}} r^{-\alpha}\mathrm{e}^{-r}\,\mathrm{d}r + \mathrm{e}^{-\delta\alpha\beta_2^{-1}}\,.$$

The right hand side is monotonously increasing in $\delta > 0$. Therefore, we may satisfy (25) for $0 < \delta \leq \beta_2 - \beta_1 \leq \beta_2$ with

$$k_{10} := \alpha^\alpha\int_0^\alpha r^{-\alpha}\mathrm{e}^{-r}\,\mathrm{d}r + \mathrm{e}^{-\alpha} - 1 \geq 0\,, \quad k_9 := 1 + \frac{(\beta_2-\beta_1)^\alpha}{\beta_2^\alpha}k_{10}\,.$$

If $k_3 = k_1\beta_1^\alpha$, $k_4 = k_2\beta_2^\alpha$ and $\ell_{11} = \ell_{12} = \ell_{21} = \ell_{22} = \ell$, condition (63) reads

$$\beta_2 - \beta_1 > \left(k_1\beta_1^\alpha + k_2k_9\beta_2^\alpha + (1 + k_9k_1k_2)\sqrt{\beta_1^\alpha\beta_2^\alpha}\right)\ell\,. \tag{64}$$

Now we assume that $\mathcal{Z}$ is a Hilbert space, A is a time-independent, selfadjoint, positive linear operator on $\mathcal{Z}$ with dense domain and compact resolvent, f is a continuous mapping from $\mathbb{R} \times \mathcal{X}$, $\mathcal{X} = D(A^\alpha)$, into $\mathcal{Z}$ satisfying a global Lipschitz condition $\|f(t,x) - f(t,y)\|_{\mathcal{Z}} \leq \ell \|x-y\|_{\mathcal{X}}$ for $x, y \in \mathcal{X}$. Let $\lambda_1 \leq \lambda_2 \leq \cdots$ denote the eigenvalues of A counted with their multiplicity and let π_1 be the orthogonal projector from $\mathcal{Z}$ onto the N-dimensional subspace spanned by the first N eigenvectors of A. Then (13) is satisfied with $k_1 = k_2 = 1$, $\beta_1 = \lambda_N$, $\beta_2 = \lambda_{N+1}$, and we find the spectral gap condition

$$\lambda_{N+1} - \lambda_N > \left(\left(\lambda_N^{\alpha/2} + \lambda_{N+1}^{\alpha/2} \right)^2 + k_{10} \frac{\lambda_N^{\alpha/2} + \lambda_{N+1}^{\alpha/2}}{\lambda_{N+1}^{\alpha/2}} (\lambda_{N+1} - \lambda_N)^\alpha \right) \ell \tag{65}$$

which holds if

$$\lambda_{N+1} - \lambda_N > 2 \left(\lambda_N^\alpha + \lambda_{N+1}^\alpha + k_{10} (\lambda_{N+1} - \lambda_N)^\alpha \right) \ell \,. \tag{66}$$

Romanov [Rom94] showed that a spectral gap condition

$$\lambda_{N+1} - \lambda_N > (\lambda_{N+1}^\alpha + \lambda_N^\alpha) \ell \tag{67}$$

is sufficient for the existence of an N-dimensional (autonomous) inertial manifold. Note that the right hand side in (66) is at most by the factor $2(1+k_{10})$ worse than the right hand side in the sharp condition (67), where $k_{10} = 0$ for $\alpha = 0$ and $k_{10} \approx 0.46$ for $\alpha = \frac{1}{2}$.

For $\alpha \leq \frac{1}{2}$, we may apply Corollary 48 which yields the strong spectral gap condition (67), too. If $\alpha \in]\frac{1}{2}, 1]$, we refer to Remark 45, which says that our approach also allows to get the sharp condition (67) in that case. Moreover, for some evolution equations it usefull to distinguish the space $\mathcal{X} = D(A^\alpha)$, in which the semiflow acts, from the space $D(A^\gamma)$ used in the Lipschitz inequality: One has to choose $\alpha \in [0, 1[$ in such a way that f is a sufficiently smooth mapping from $\mathbb{R} \times \mathcal{X}$ as required for the existence theory. However it is possible to satisfy and to require a Lipschitz inequality $\|f(x) - f(y)\|_{\mathcal{Z}} \leq \nu(x-y)$ for $x, y \in D(A)$ with $\gamma \in [0, \min\{\alpha, \frac{1}{2}\}[$ and some norm ν on $D(A^\gamma)$. Especially, for $\nu(x) = \ell |x|_{D(A^\gamma)}$, our Corollary 48 yields a spectral gap condition

$$\lambda_{N+1} - \lambda_N > (\lambda_{N+1}^\gamma + \lambda_N^\gamma) \ell$$

which is weaker than (67) if $\gamma < \alpha$. As an concrete application we consider a reaction-diffussion equation

$$u_t = u_{\xi\xi} + F(\xi, u, \nabla u) \,, \quad u(t,0) = u(t,1) = 0$$

with

$$|F(\xi, u, v) - F(\xi, u', v')| \leq \ell_0 |u - u'| + \ell_1 |v - v'|$$

in $\mathcal{Z} = L_2([0,1])$ as studied by P. Brunovský and I. Teresščák, [BT91]. Here $-A$ is the Laplacian with Dirichlet boundary condition on $[0,1]$, and $f(t,x)(\xi) =$

$F(t,\xi,x(\xi),\nabla x(\xi))$. For the existence theory we need $\alpha > \frac{3}{4}$, but it is possible to choose $\nu(x) = \sqrt{2}\ell_0|x|+\sqrt{2}\ell_1|x|_{\frac{1}{2}}$ in order to cover the gradient in the nonlinearity. Corollary 48 yields the spectral gap condition

$$\lambda_{N+1} - \lambda_N > \sqrt{2}(2\ell_0 + \ell_1(\lambda_N^{\frac{1}{2}} + \lambda_{N+1}^{\frac{1}{2}}))\,,$$

i.e.

$$(2N+1)\pi^2 > 2\sqrt{2}\ell_0 + \sqrt{2}(2N+1)\pi\ell_1$$

which is weaker than the spectral gap condition found in [BT91].

Now let

$$\psi(t) = \beta_2^{-\alpha}t^{-\alpha} + 1$$

as in [Tem97]. Then $t_* = \infty$, $\psi_* = 1$ and we may choose

$$k_{10} := \Gamma(1-\alpha)\,, \quad k_9 := 1 + k_{10}$$

to satisfy (25). In the special case $\ell_1 = \ell_2 = \ell$, $k_3 = k_1\beta_1^\alpha$, $k_4 = k_2\beta_2^\alpha$, condition (63) reads now

$$\beta_2 - \beta_1 > \Big(k_1\beta_1^\alpha + (1 + k_1k_2(1+k_{10}))\sqrt{\beta_1^\alpha\beta_2^\alpha} + k_2(1+k_{10})\beta_2^\alpha\Big)\,\ell\,. \tag{68}$$

For a Banach space $\mathcal{Z}$, a time-independent, sectorial linear operator A on $\mathcal{Z}$ with dense domain $D(A)$, and a time-independent, continuous mapping f from $\mathcal{X} = D(A^\alpha)$ into $\mathcal{Z}$ satisfying a global Lipschitz condition $\|f(x) - f(y)\|_{\mathcal{Z}} \leq \ell\|x-y\|_{\mathcal{X}}$ where $\mathcal{X} = D((A+a)^\alpha)$ with fixed $a \in \mathbb{R}$, $\alpha \in [0,1[$ with $\Re\sigma(A)+a > 0$, and under assumption (13) with $\psi(t) = \beta_2^{-\alpha}t^{-\alpha} + 1$, Temam showed ([Tem97], Theorem IX.2.1) that there are constants c_1 and c_2 independent of the Lipschitz constant ℓ and the boundedness constant ℓ_0 of the nonlinearity f, such that the spectral gap condition

$$\beta_2 - \beta_1 \geq c_1(\ell_0 + \ell + \ell^2)(\beta_2^\alpha + \beta_1^\alpha)\,, \quad \beta_1^{1-\alpha} \geq c_2(\ell_0 + \ell) \tag{69}$$

implies the existence of an autonomous inertial manifold in the autonomous case.

Note that our condition (68) is of similar form as (69) but (68) contains only known constants and is applicable for the nonautonomous case, too. Moreover, in contrast to (69), in our condition (68), the right hand side is linear in the Lipschitz constant ℓ.

Finally let

$$\psi(t) = \alpha^\alpha\beta_2^{-\alpha}t^{-\alpha} + 1\,.$$

Then we choose $t_* := \infty$ and have $\psi_* = 1$. Since

$$\begin{aligned}\delta\int_0^{t_*}\psi(\tau)\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau &= \delta\beta_2^{-\alpha}\int_0^\infty(\alpha^\alpha\tau^{-\alpha} + \beta_2^\alpha)\mathrm{e}^{-\delta\tau}\,\mathrm{d}\tau\\ &= \alpha^\alpha\delta^\alpha\beta_2^{-\alpha}\Gamma(1-\alpha) + 1\,,\end{aligned}$$

for $\delta > 0$, we may choose

$$k_{10} := \alpha^\alpha \Gamma(1-\alpha)\,, \quad k_9 := 1 + \frac{(\beta_2-\beta_1)^\alpha}{\beta_2^\alpha} k_{10} \tag{70}$$

in order to satisfy (25) for $\delta \in]0, \beta_2 - \beta_1[$. In the special case $\ell_1 = \ell_2 = \ell$, $k_3 = k_1\beta_1^\alpha$, $k_4 = k_2\beta_2^\alpha$, condition (63) takes the form (64).

Whilst we are yet not in a position to deal with retardation or stochastic perturbation, we try to compare our result with that one found by L. Boutet de Monvel, I.D. Chueshov and A.V. Rezounenko in [BdMCR98] and by I.D. Chueshov, M. Scheutzow in [CS01] for the special case of a semilinear parabolic equation without perturbation and without retardation. There $\mathcal{Z}$ is a Hilbert space, A is a time-independent, selfadjoint, positive linear operator on $\mathcal{Z}$ with dense domain and compact resolvent, f is a continuous mapping from $\mathbb{R} \times \mathcal{X}$, $\mathcal{X} = D(A^\alpha)$, into $\mathcal{Z}$ satisfying a global Lipschitz condition $\|f(t,x) - f(t,y)\|_{\mathcal{Z}} \le \ell\|x-y\|_{\mathcal{X}}$. Let $\lambda_1 \le \lambda_2 \le \cdots$ denote the eigenvalues of A counted with their multiplicity and let π_1 be the orthogonal projector from $\mathcal{Z}$ onto the N-dimensional subspace spanned by the first N eigenvectors of A. Chueshov and Scheutzow [CS01] found the spectral gap condition

$$\lambda_{N+1} - \lambda_N > 2\left(\lambda_N^\alpha + \lambda_{N+1}^\alpha + \alpha^\alpha\Gamma(1-\alpha)(\lambda_{N+1}-\lambda_N)^\alpha\right)\ell\,, \tag{71}$$

and Boutet de Monvel, Chueshov and Rezounenko found

$$\lambda_{N+1} - \lambda_N \ge 4\left(\lambda_N^\alpha + \lambda_{N+1}^\alpha + \alpha^\alpha\Gamma(1-\alpha)(\lambda_{N+1}-\lambda_N)^\alpha\right)\ell\,,$$

which is is little bit worse than (71).

In this situation the exponential dichotomy condition (13) is satisfied with $k_1 = k_2 = 1$, $\beta_1 = \lambda_N$, $\beta_2 = \lambda_{N+1}$, $k_3 = \lambda_N^\alpha$, $k_4 = \lambda_{N+1}^\alpha$, and we find again the spectral gap condition (65) but here with k_{10} given by (70). Obviously our condition (65) is a little weaker than (71). Note again that Corollary 48 would only require the spectral gap condition 67. So we have good chances to extend our result to retarded semilinear parabolic equations and, possibly, to semilinear parabolic equations with stochastic perturbation.

Summarizing the examples above, one can see that at least in these examples our approach allows to get the same or weaker spectral gap conditions.

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