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Stability Theorems for Nonlinear Functional Differential Equations

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Abstract. New approach in stability theory for a class of retarded nonlinear functional differential equations is discussed. The problem of stability of the zero solution is considered under assumption that the system of interest has a trivial linearization, i.e. it is essentially nonlinear. Sufficient conditions for uniform asymptotic stability and instability are given by auxiliary functionals of Lyapunov-Krasovskii type. The method is also applicable to linear systems with a small parameter in standard form. Some examples are given.

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1 Introduction

The paper is devoted to the problem of stability of the zero solution of nonlinear system of functional differential equations (FDE) of retarded type

$$\dot{x} = F(t, x_t). \tag{1}$$

There is no need to elaborate on the role that Lyapunov-Krasovskii functionals play in the analysis of asymptotic properties of solutions of FDE. In particular, a suitable functional may ensure the uniform asymptotic stability of a trivial solution of (1). Due to classical results [1], [2]. suitable here may mean uniformly positive definite on state space $C_h = C([-h, 0]), \mathbf{R}^n)$, which strictly and uniformly decreases along nontrivial solutions. There is the celebrated converse theorem on Lyapunov-Krasovskii functional [2]. But to find actually such functionals in concrete examples is not an easy problem. In this paper we establish new sufficient conditions on stability for a class of FDE in terms of functionals which satisfy less strong restrictions. We suggest a new approach in context of generalized Lyapunov's direct method [3] – [6]. Unlike the well-known theorems on stability for FDE [1], [2] suitable functionals satisfy main restrictions only in

This is the preliminary version of the paper.

some cone $\mathcal{A}_R^h \subset C_h$ not in the whole space C_h, moreover, they are nonmonotone along nontrivial solutions of (1). It gives a possibility to use a simple procedure to construct suitable functionals.

The paper is organized as follows. In the second section we give the statement of the problem, some definitions and mathematical facts. Theorems on uniform asymptotic stability and instability are stated and proved in sections 3 and 4, respectively. In the last section we consider two examples. The first one is an example of nonlinear scalar equation with deviating argument which has unstable zero solution for all values of the constant delay $h \in [a, b]$ but, from the other hand, it is shown that there exist a time-varying delay $h_0(t)$ with the same range of values, $h_0(t) \in [a, b]$ for all t, such that this equation with delay $h_0(t)$ already have uniformly asymptotically stable trivial solution.

We also study the parametric resonance in linear equation with small parameter of Mathieu type. It is shown that the delay being introduced may damp the demultiplicative parametric resonances and make the equation either unstable or asymptotically stable.

2 Preliminaries

Consider a system of nonlinear functional differential equations with finite delay written as

$$\dot{x}(t) = F(t, x_t), \tag{2}$$

where $F : G_H^h \to \mathbf{R}^n$, $G_H^h = \mathbf{R}_+ \times \Omega_H^h$, $\mathbf{R}_+ = [0, \infty)$, $\Omega_H^h = \{\varphi \in C_h : \|\varphi\| < H\}$ is the open $H-$ ball in the Banach space $C_h = C([-h, 0], \mathbf{R}^n)$ of continuous functions $\varphi : [-h, 0] \to \mathbf{R}^n$ with the supremum norm $\|\varphi\| = \max\{|\varphi(s)| \colon -h \leq s \leq 0\}$, $|\cdot|$ is a norm in $\mathbf{R}^n$. For a given function $x(t)$ we denote by x_t the element in C_h defined by $x_t(s) = x(t + s)$, $-h \leq s \leq 0$. In the context of FDE the element x_t is called the *state* at time t.

Denote by $\mathcal{UI}(\mathbf{R}_+)$ a set of all functions $L : \mathbf{R}_+ \to \mathbf{R}_+$ which are integrable on any finite segment $[t_0, t_0 + \Delta] \subset \mathbf{R}_+$ and for any $\Delta > 0$ there exists a constant $L_\Delta > 0$ such that

$$\int_{t_0}^{t_0+\Delta} L(t)\, dt \leq L_\Delta \qquad \text{for any } t_0 \in \mathbf{R}_+. \tag{3}$$

We assume that there are exist functions $L, M^0 \in \mathcal{UI}(\mathbf{R}_+)$ and a constant $d_0 > 1$ such that for any $t \in \mathbf{R}_+$ and $\varphi, \psi \in \Omega_H^h$

$$|F(t, \varphi) - F(t, \psi)| \leq L(t)\|\varphi - \psi\|, \tag{4}$$

$$|F(t, \varphi)| \leq M^0(t)\|\varphi\|^{d_0} \tag{5}$$

for any $t \in \mathbf{R}_+$ and $\varphi, \psi \in \Omega_H^h$.

A solution of (2) through $(t_0, \varphi) \in \mathbf{R}_+ \times \Omega_H^h$ will be denoted by $x(t_0, \varphi) : \mathbf{R}_+ \to \mathbf{R}^n, t \mapsto x(t; t_0, \varphi)$, so that $x_{t_0}(t_0, \varphi) = \varphi$. It is known that $x(t_0, \varphi)$ satisfies the integral equation

$$x(t; t_0, \varphi) = \varphi(0) + \int_{t_0}^{t} F(\tau, x_\tau)\, d\tau, \quad x_{t_0} = \varphi, \quad t \geq t_0. \tag{6}$$

Using this representation and Gronwall's lemma it is easy to get the following results [7].

Lemma 1. *Let $t_0 \in \mathbf{R}_+$ and $\varphi \in \Omega_H^h$ be given and the functional F satisfies Lipschitz inequality (4). Then until $x(t_0 + \Delta; t_0, \varphi) \in \Omega_h^h$ the following inequality holds*

$$\|x_{t_0+\Delta}(t_0, \varphi)\| \leq \|\varphi\| \exp(L_\Delta), \tag{7}$$

where L_Δ is a constant from the estimate (3).

Note that the right-hand part of inequality (7) does not depend on t_0.

Lemma 2. *Let $t_0 \in \mathbf{R}_+$ and $\varphi \in \Omega_H^h$ be given and the functional F satisfies inequalities (4) and (5) then*

$$|x(t_0 + \Delta; t_0, \varphi) - \varphi(0)| \leq \|\varphi\|^{d_0} E_\Delta, \tag{8}$$

where $E_\Delta = M_\Delta^0 \exp(d_0 L_\Delta)$, M_Δ^0 and L_Δ are the constants from the estimates of the type (3) for functions $M^0(t)$ and $L(t)$ respectively.

Lemma 3. *Let $x : [t_0 - h, \infty) \to \mathbf{R}^n$ be a continuous function and there exist a constant $R > 1$ such that $|x(t)| \leq \|x_t\|/R \equiv (1/R) \max\{|x(t+s)| : -h \leq s \leq 0\}$ for all $t \geq t_0$.*

Then $\lim_{t\to\infty} |x(t)| = 0$, and

$$|x(t)| \leq \|x_{t_0}\|/R^{N+1} \qquad \text{for } t \geq t_0 + Nh, N = 0, 1, 2, \ldots .$$

In this paper we use the known definition of stability.

Definition 4. The zero solution $x = 0$ of the system (2) is said to be

stable if for each $\sigma \geq 0$, $\alpha > 0$ there is $\beta = \beta(\alpha, \sigma) > 0$ such that $\varphi \in \Omega_\beta^h$ implies that $x_t(\sigma, \varphi) \in \Omega_\alpha^h$ for any $t \geq \sigma$;

uniformly stable if it is stable and β is independent of σ;

asymptotically stable, if it is stable and for each $\sigma \geq 0$ there is $\beta_0 = \beta_0(\sigma) > 0$ such that $\varphi \in \Omega_{\beta_0}^h$ implies $x_t(t; \sigma, \varphi) \to 0$ as $t \to \infty$;

uniformly asymptotically stable if it is uniformly stable and if there is a $\beta_0 > 0$ and for each $\eta > 0$ there exists $t_0(\eta) > 0$ such that for any $\sigma \geq 0$ $x_\sigma = \varphi \in \Omega_{\beta_0}^h$ implies $x_t(\sigma, \varphi) \in \Omega_\eta^h$ for $t \geq \sigma + t_0(\eta)$.

For given $h_0 > h$ and $R \geq 1$ consider the set

$$\mathcal{A}_R^{h_0} = \{\varphi \in C_{h_0} : \|\varphi\| \leq R|\varphi(0)|\}.$$

It is easy to see that
$\mathcal{A}_R^{h_0} \neq \emptyset$ for $R > 1$,
$\mathcal{A}_{R_1}^{h_0} \subset \mathcal{A}_{R_2}^{h_0}$ for $R_1 < R_2$,
the boundary $\partial\mathcal{A}_R^{h_0}$ of the set $\mathcal{A}_R^{h_0}$ is defined as

$$\partial\mathcal{A}_R^{h_0} = \{\varphi \in C_{h_0} : \|\varphi\| = R|\varphi(0)|\},$$

$\mathcal{A}_1^{h_0} \equiv \partial\mathcal{A}_1^{h_0}$ and $\mathcal{A}_1^{h_0} \subset \mathcal{A}_R^{h_0}$ for any $R > 1$,
$\mathcal{A}_R^{h_0}$ is a nonconvex cone in C_{h_0}.

The cone $\mathcal{A}_R^{h_0}$ plays a crucial role in our approach. The fact is that the norm $\|x_t(\sigma,\varphi)\|$ may increase if and only if $x_t(\sigma,\varphi) \in \mathcal{A}_1^{h_0}$ and $|x(t;\sigma,\varphi)|$ tends to zero when $x_t(\sigma,\varphi) \notin \mathcal{A}_R^{h_0}$ for some $R > 1$. Therefore in the context of the stability problem it is enough to investigate a behavior of the state $x_t(\sigma,\varphi)$ only in the cone $A_R^{h_0}$ not in the whole neighborhood $\Omega_H^{h_0}$.

3 Sufficient conditions on asymptotic stability

In this section we present sufficient conditions for uniform asymptotic stability of the zero solution of the system (2) in terms of Lyapunov's functionals $v(t,\varphi)$ which can be nonmonotone along the solutions. It means that the derivative $\dot{v}|_{(2)}(\sigma,\varphi)$ of the functional v along the solution of (2) can change the sign. This derivative is defined as

$$\dot{v}|_{(2)}(\sigma,\varphi) = \lim_{\Delta t \to +0} \frac{v(\sigma+\Delta t, x_{\sigma+\Delta t}(\sigma,\varphi)) - v(\sigma,\varphi)}{\Delta t}.$$

If v is differentiable $\dot{v}|_{(2)}(\sigma,\varphi)$ is obtained using the chain rule.

We start with the following technical lemma.

Lemma 5. *Let $h_0 \geq h$ and $R > 1$ be given. Assume that for some $\tau_0 \geq 0$ and $\psi_0 \in \mathcal{A}_R^{h_0} \cap \Omega_\eta^{h_0}$ a solution $x(\tau_0,\psi_0)$ of the system (2) is defined for $\tau_0 \leq t \leq \tau_0 + 2h_0$ and $\eta < \eta_R$,*

$$\eta_R = \left[\frac{R-1}{(R+1)R^{d_0}E_0}\right]^{1/(d_0-1)}, \quad E_0 = M_{2h_0}^0 \exp(d_0 L_{2h_0}). \tag{9}$$

Then $x_t(\tau_0,\psi_0) \in \mathcal{A}_R^{h_0}$ for $\tau_0 + h_0 \leq t \leq \tau_0 + 2h_0$.

If, in addition, $\|x_t(\tau_0,\psi_0)\| < \eta_R$ for all $t \geq \tau_0 + h_0$, then $x_t(\tau_0,\psi_0) \in \mathcal{A}_R^{h_0}$ for all $t \geq \tau_0 + h_0$.

Proof. According to Lemma 2 the solution $x(\tau_0,\psi_0)$ satisfies the inequality

$$|x(t;\tau_0,\psi_0) - x(\tau_0)| \leq \|x_{\tau_0}\|^{d_0} E_0 \tag{10}$$

for any $t \in [\tau_0, \tau_0 + 2h_0]$, here $x_{\tau_0} = \psi_0, x(\tau_0) = \psi_0(0)$. Using (10), we obtain the following upper and lower estimates for the norm $|x(t; \tau_0, \psi_0)|$ on the segment $[\tau_0, \tau_0 + 2h_0]$:

$$|x(t;\tau_0,\psi_0)| \le |x(t;\tau_0,\psi_0) - x(\tau_0) + x(\tau_0)| \le |x(\tau_0)| + \|x_{\tau_0}\|^{d_0} E_0 \\ \le |x(\tau_0)|(1 + R^{d_0}|x_{\tau_0}|^{d_0-1}E_0), \qquad (11)$$

$$|x(t;\tau_0,\psi_0)| \ge |x(\tau_0)| - \|x_{\tau_0}\|^{d_0} E_0 \ge |x(\tau_0)|(1 - R^{d_0}|x_{\tau_0}|^{d_0-1}E_0). \qquad (12)$$

Hence for $t \in [\tau_0 + h_0, \tau_0 + 2h_0]$ we have

$$\frac{\|x_t\|}{|x(t)|} \le \frac{\max\{|x(t+s), t - h_0 \le s \le t\}}{\min\{|x(s)|, \tau_0 + h_0 \le s \le \tau_0 + 2h_0\}} \le \frac{1 + R^{d_0}E_0|x(\tau_0)|^{d_0-1}}{1 - R^{d_0}E_0|x(\tau_0)|^{d_0-1}} \\ \le \frac{1 + R^{d_0}E_0\eta^{d_0-1}}{1 - R^{d_0}E_0\eta^{d_0-1}} < \frac{1 + R^{d_0}E_0\eta_R^{d_0-1}}{1 - R^{d_0}E_0\eta_R^{d_0-1}} = R. \quad (13)$$

It means, that $\|x_t\| < R|x(t)|$, i.e. $x_t \in \mathcal{A}_R^{h_0}$, for $t \in [\tau_0 + h_0, \tau_0 + 2h_0]$.

If $\|x_t(\tau_0, \psi_0)\| < \eta_R$ for all $t \ge \tau_0 + h_0$, then $|x(\tau_0 + kh_0; \sigma, \psi_0)| < \eta_R$ for any $k = 1, 2, \ldots$. The constant E_0 does not depend on x_{τ_0}, consequently, the estimates (11)–(13) hold for $\tau_0 + kh_0 \le t \le \tau_0 + (k+1)h_0, \quad k = 1, 2, \ldots$. It means that $x_t \in \mathcal{A}_R^{h_0}$ for any $t \ge \tau_0$. Lemma is proved.

Let $\mathcal{K}$ denotes the *Hahn class*, i.e. the set of all continuous strictly increasing functions $a : \mathbf{R}_+ \to \mathbf{R}_+$ such that $a(0) = 0$.

Theorem 6. *Suppose that for some $h_0 \ge h$ and $R > 1$ the following assumptions hold:*

1) *there exist functionals $v, \Phi : G_H^{h_0} \to \mathbf{R}$ and functions $a, b \in \mathcal{K}$ such that*
   a) $\dot v|_{(2)}(t,\varphi) \le \Phi(t,\varphi)$,
   b) $v(t,\varphi) \le b(\|\varphi\|)$ *for* $(t,\varphi) \in G_H^{h_0}$,
   c) $v(t,\varphi) \ge a(\|\varphi\|)$ *for* $t \ge 0$ *and* $\varphi \in \mathcal{A}_R^{h_0} \cap \Omega_H^{h_0}$;

2) *there exist constants $d > 1, m > 0$ and a function $M \in \mathcal{UI}(\mathbf{R}_+)$ such that*

$$|\Phi(t,\varphi)| \le M(t)\|\varphi\|^{d_0} \quad \text{for } (t,\varphi) \in G_H^{h_0},$$

$$|\Phi(t,\varphi) - \Phi(t,\psi)| \le M(t) r^{d-1}\|\varphi - \psi\|$$

for $\forall t \ge 0$ *and* $\varphi, \psi \in \Omega_r^{h_0}$, $0 < r < H$;

3) *there exist constants $T > 0, \beta > 0$ and $\delta > 0$ such that for any $t_0 \ge 0$, $x_0 \in B_\beta$ and $\Delta t \ge T$*

$$\mathcal{I}(\Delta t, t_0, x_0) = \int_{t_0}^{t_0+\Delta t} \Phi(t, x_0)\, dt \le -2\delta |x_0|^d \Delta t.$$

Then the zero solution of (1) is uniformly asymptotically stable.

Proof. First of all we will show that $x_t(t_0, x_{t_0}) \in \mathcal{A}_R^{h_0}$ for all $t \geq 0$ if $x_{t_0} \in \mathcal{A}_R^{h_0}$ at some moment t_0 and $\|x_{t_0}\|$ is small enough. Then we prove that the zero solution of the system (2) is uniformly stable and $\|x_t\| \to 0$ as $t \to \infty$. Uniformity of the asymptotic stability is guaranteed by the properties of the functional v in the cone $\mathcal{A}_R^{h_0}$.

Let us fix an arbitrary small $\varepsilon \in (0, \eta_R)$, where η_R is defined as in Lemma 5 by given $h_0 \geq h$ and $R > 1$. Put $\varepsilon_1 = \varepsilon/2$. Denote by

$$\{v < \gamma\}_\tau = \{\varphi \in \Omega_H^{h_0} : v(\tau, \varphi) < \gamma\}$$

a cut for $t = \tau$ of the region

$$\{v < \gamma\} = \{(t, \varphi) \in G_H^{h_0} : v(t, \varphi) < \gamma\}.$$

Due to the positive definiteness of the functional v in the region $\mathbf{R}_+ \times (\Omega_H^{h_0} \cap \mathcal{A}_R^{h_0})$ there exists a constant $\gamma > 0$ such that for all $t \geq 0$

$$\{v < \gamma\}_t \cap \mathcal{A}_R^{h_0} \subset \Omega_{\varepsilon_1}^{h_0}.$$

It is enough to take $\gamma = a(\varepsilon_1)$. Choose a value $\eta_0 > 0$ such that

$$\eta_0 < \eta_0(1 + \eta_0^{d_0 - 1} E_{h_0}) < b^{-1}(\gamma), \tag{14}$$

where $E_{h_0} = M_{h_0}^0 \exp[L_{h_0}(d_0 + 1)]$. Note that (14) implies that $\eta_0 < \eta_R$.

Let $\sigma \geq 0$ and $\psi \in \Omega_{\eta_0}^{h_0}$ be given. Inequality (14) implies that $v(\sigma, \psi) < \gamma$. If $\psi \notin \mathcal{A}_R^{h_0}$, then $\|x_t(\sigma, \psi)\|$ will decrease while $x_t \notin \mathcal{A}_R^{h_0}$. The rate of decreasing is given by Lemma 3. Suppose that $x_{\tau_0} \in \mathcal{A}_R^{h_0}$ for some $\tau_0 \geq \sigma$. According to the choice of η_0 (see the inequality (14)) $v(t, x_t) < \gamma$ for $t \in [\tau_0, \tau_0 + h_0]$. Due to Lemma 5 $x_{\tau_0 + h_0} \in \mathcal{A}_R^{h_0}$, hence,

$$a(\|x_{\tau_0 + h_0}\|) \leq v(\tau_0 + h_0, x_{\tau_0 + h_0}) < \gamma = a(\varepsilon_1),$$

therefore $\|x_{\tau_0 + h_0}\| < \varepsilon_1 < \eta_R$ and $x_t \in \mathcal{A}_R^{h_0}$ at least for $t \in [\tau_0 + h_0, \tau_0 + 3h_0]$. $x_t(\sigma, \psi)$ will be in $\mathcal{A}_R^{h_0}$ while $\|x_t\| < \eta_R$. Remember that $\|x_t\|$ may increase only if $x_t \in \mathcal{A}_1^{h_0} \subset \mathcal{A}_R^{h_0}$. According to the choice of γ we see that $\|x_t\| \leq \eta_R/2$ till $v(t, x_t) \leq \gamma$. Suppose that $v(t_0, x_{t_0}) = \gamma$ for some $t_0 \geq \tau_0 + h_0$ the point x_t leaves the domain $\{v < \gamma\}$. Note that $x_{t_0} \in \mathcal{A}_R^{h_0}$, i.e. $\|x_{t_0}\| \leq R|x(t_0)|$. According to condition 2a) of the theorem

$$\dot{v}|_{(2)}(t, x_t) \leq \Phi(t, x_t).$$

Thus

$$v(t, x_t(\sigma, \psi)) - v(t_0, x_{t_0}(\sigma, \psi)) \leq \int_{t_0}^{t} \Phi(\tau, x_\tau)\, d\tau. \tag{15}$$

Define a function $z : \mathbf{R} \to \mathbf{R}^n$ such that $z(t) = x(t; \sigma, \psi)$ for $t_0 - h_0 \leq t \leq t_0$ and $z(t) = x(t_0)$ for $t \geq t_0$ Adding and subtracting $\int_{t_0}^{t} \Phi(\tau, z_\tau)\, d\tau$, we get

$$\int_{t_0}^{t} \Phi(\tau, x_\tau)\, d\tau = \int_{t_0}^{t} \Phi(\tau, z_\tau)\, d\tau + \int_{t_0}^{t} [\Phi(\tau, x_\tau) - \Phi(\tau, z_\tau)]\, d\tau. \tag{16}$$

According to Lemma 2
$$\|x_\tau - z_\tau\| \le M_1 \|x_{t_0}\|^{d_0},$$
for $\tau \in [t_0, t_0 + T_1]$, where the constant $T_1 \ge T$ will be selected below, the constant M_1 depends only on T_1. Using Lemma 1 and the Lipschitz condition we obtain

$$\int_{t_0}^{t_0+T_1} [\Phi(\tau, x_\tau) - \Phi(\tau, z_\tau)]\, d\tau \le C_0 T_1 \|x_{t_0}\|^{d+d_0-1}. \tag{17}$$

The estimate (17) is uniform with respect to $t_0 \ge 0$ and $x_{t_0} \in \Omega_H^{h_0}$, i.e. the constant $C_0 > 0$ depends only on T_1.

To estimate first integral in the right-hand side of (16) we represent it in the form

$$\int_{t_0}^{t} \Phi(\tau, z_\tau)\, d\tau = \int_{t_0}^{t_0+h_0} \Phi(\tau, z_\tau)\, d\tau + \int_{t_0}^{t} \Phi(\tau, y_\tau)\, d\tau - \int_{t_0}^{t_0+h_0} \Phi(\tau, x(t_0))\, d\tau, \tag{18}$$

Due to construction of the function z, condition 2) of the theorem and Lemma 1, we obtain

$$\left| \int_{t_0}^{t_0+h_0} \Phi(\tau, z_\tau)\, d\tau \right| \le h_0 M_{h_0} \|x_{t_0}\|^d \exp(d L_{h_0}), \tag{19}$$

$$\left| \int_{t_0}^{t_0+h_0} \Phi(\tau, y_\tau)\, d\tau \right| \le h_0 M_{h_0} |x(t_0)|^d \exp(d L_{h_0}). \tag{20}$$

Choose $T_1 \ge T$ such that

$$M_{h_0}(R^d + 1) h_0 \exp(d L_{h_0}) / T_1 \le \delta/2, \tag{21}$$

where δ is a constant from condition 3) of the theorem. Using condition 3) and taking into account that $\|x_{t_0}\| < R|x(t_0)|$, from (18)–(21) we get

$$\begin{aligned}
\int_{t_0}^{t_0+T_1} \Phi(\tau, z_\tau)\, d\tau &= \int_{t_0}^{t_0+T_1} \Phi(\tau, x(t_0))\, d\tau - \int_{t_0}^{t_0+h_0} \Phi(\tau, x(t_0))\, d\tau + \int_{t_0}^{t_0+h_0} \Phi(\tau, z_\tau)\, d\tau \\
&\le \int_{t_0}^{t_0+T_1} \Phi(\tau, x(t_0))\, d\tau + \left| \int_{t_0}^{t_0+h_0} \Phi(\tau, x(t_0))\, d\tau \right| + \left| \int_{t_0}^{t_0+h_0} \Phi(\tau, z_\tau)\, d\tau \right| \\
&\le -2\delta |x(t_0)|^d T_1 + h_0 M_{h_0} \exp(d L_{h_0}) (|x(t_0)|^d + \|x_{t_0}\|^d) \\
&\le |x(t_0)|^d T_1 \left(-2\delta + \frac{M_{h_0}(R^d + 1) h_0 \exp(d L_{h_0})}{T_1} \right) \le -\frac{3}{2} \delta |x(t_0)|^d T_1.
\end{aligned}$$

Thus

$$\int_{t_0}^{t_0+T_1} \Phi(\tau, z_\tau)\, d\tau \leq -\frac{3}{2}\delta |x(t_0)|^d T_1. \tag{22}$$

Suppose that ε is small enough to be true the following inequality

$$C_0 R^{d+d_0-1} (\varepsilon/2)^{d_0-1} \leq \delta/2. \tag{23}$$

Then from (15)–(17) and (22) we have

$$\begin{aligned} v(t_0+T_1, x_{t_0+T_1}) &\leq v(t_0, x_{t_0}) + \int_{t_0}^{t} \Phi(\tau, z_\tau)\, d\tau + \int_{t_0}^{t} [\Phi(\tau, x_\tau) - \Phi(\tau, z_\tau)]\, d\tau \\ &\leq v(t_0, x_{t_0}) - \frac{3}{2}\delta |x(t_0)|^d T_1 + C_0 T_1 R^{d+d_0-1} |x(t_0)|^{d+d_0-1} \\ &\leq v(t_0, x_{t_0}) + |x(t_0)|^d T_1 (-\frac{3}{2}\delta + C_0 T_1 R^{d+d_0-1} |x(t_0)|^{d_0-1}) \\ &\leq v(t_0, x_{t_0}) - \delta |x(t_0)|^d T_1 < v(t_0, x_{t_0}). \end{aligned}$$

It gives us the main inequality

$$v(t_0+T_1, x_{t_0+T_1}) \leq v(t_0, x_{t_0}) - \delta |x(t_0)|^d T_1. \tag{24}$$

We emphasize that the estimate (24) is uniform with respect to $t_0 \geq 0$ and $x_{t_0} \in \mathcal{A}_R^{h_0} \cap \Omega_{\varepsilon_1}^{h_0}$. Inequality (24) means that the state x_t has returned into the domain $\{v < \gamma\}_t$, moreover, we can choose $\varepsilon > 0$ so small that x_t does not leave the ball $\Omega_\varepsilon^{h_0}$. Indeed, according to Lemma 2 for $t \in [t_0, t_0 + T_1]$

$$|x(t; t_0, x_{t_0}) - x(t_0)| \leq \|x_{t_0}\|^{d_0} M_{T_1} \exp(d_0 L_{T_1}) \leq |x_{t_0}|^{d_0} R^{d_0} M_{T_1} \exp(d_0 L_{T_1}),$$

therefore

$$|x(t; t_0, x_{t_0})| \leq |x(t_0)| + |x(t) - x(t_0)| \leq \varepsilon,$$

if ε is small enough to ensure

$$(\varepsilon/2)^{d_0} R^{d_0} M_{T_1} \exp(d_0 L_{T_1}) \leq \varepsilon/2.$$

According to Lemma 5 $x_t \in \mathcal{A}_R^{h_0}$ for $t \in [t_0, t_0 + T_1]$. It has been shown that at the moment $t_1 = t_0 + T_1$ x_{t_1} belongs to the domain $\{v < \gamma\}_{t_1} \subset \Omega_{\varepsilon_1}^{h_0}$. If the point x_t will leave the domain $\{v < \gamma\}_t$ again at some moment $t'_0 > t_1$ then it will return back in finite time less than T_1 because all estimates we employed above to obtain the main inequality are uniform with respect to $t \geq t_0$ and $x_{t_0} \in \mathcal{A}_R^{h_0} \cap \Omega_{\varepsilon_1}^{h_0}$. Consequently, we have proved that $x_t \in \mathcal{A}_R^{h_0} \cap \Omega_{\varepsilon_1}^{h_0}$ for all $t \geq t_0$.

Since ε is arbitrary small and η_0 does not depend on initial moment σ, the uniform stability of the zero solution of the system (2) is proved.

To prove the asymptotic stability we note that in virtue of the uniformity of all estimates derived above with respect to $t \geq t_0$ and $x_{t_0} \in \mathcal{A}_R^{h_0} \cap \Omega_{\varepsilon_1}^{h_0}$ (24) yields

$$0 < v(t_k + T_1, x_{t_k+T_1}) \leq v(t_0, x_{t_0}) - \delta T_1(|x(t_0)|^d + |x(t_1)|^d + \ldots + |x(t_k)|^d), \tag{25}$$

where $t_k = t_0 + kT_1$, for any integer $k \geq 1$. By condition 1c) of the theorem $v(t_0, x_{t_0}) \geq a(\|x_{t_0}\|) > 0$, and (25) means that $|x(t_k)| \to 0$ as $k \to \infty$, therefore $\|x_{t_k}\| \to 0$ as $k \to \infty$, because $\|x_{t_k}\| < R|x(t_k)|$.

Uniformity of the asymptotic stability follows from Lemma 3 if $x_t \notin \mathcal{A}_R^{h_0}$ and from conditions 1b) and 1c) if $x_t \in \mathcal{A}_R^{h_0}$. The theorem is proved.

4 Sufficient conditions for instability

Theorem 7. *Suppose that for some $h_0 \geq h$, $\beta > 0$, $\sigma > 0$ and $R > 1$ there exist functionals $v, \Phi : [\sigma, \infty) \times \Omega_\beta^{h_0} \to \mathbf{R}$ such that the following conditions satisfied:*

1) $\dot{v}|_{(1)}(t, \varphi) \geq \Phi(t, \varphi)$ *for* $(t, \varphi) \in [\sigma, \infty) \times \Omega_\beta^{h_0}$;
2) *for each* $t \geq \sigma$ *and* $\eta, 0 < \eta < \beta$, *there exists* $\varphi \in \mathcal{A}_R^{h_0} \cap \Omega_\eta^{h_0}$ *such that* $v(t, \varphi) > 0$;
3) *there exists a function* $b \in \mathcal{K}$ *such that* $v(t, \varphi) \leq b(\|\varphi\|)$ *for each* $(t, \varphi) \in [\sigma, \infty) \times \mathcal{A}_R^{h_0} \cap \Omega_\beta^{h_0}$;
4) *the functional* Φ *satisfies condition 2) of Theorem 6 for* $(t, \varphi) \in [\sigma, \infty) \times \Omega_\beta^{h_0}$ *and* $0 < r < \beta$;
5) *there exist constants* $T > 0$ *and* $\delta > 0$ *such that for any* $t_0 \geq \sigma$, $x_0 \in B_\beta$ *and* $\Delta t \geq T$
$$\mathcal{I}(\Delta t, t_0, x_0) \geq 2\delta |x_0|^d \Delta t.$$

Then the zero solution of the system (1) *is unstable.*

Proof. By way of contradiction, assume that the zero solution of (2) is stable. By the conditions of the theorem choose a small enough value $\varepsilon < \min\{\beta, \eta_R\}$ and a constant $T_1 \geq h_0$ such that the inequalities (21) and (23) hold. According to our assumption there exists $\eta_0 > 0$ such that for any initial function $\psi_0 \in \Omega_{\eta_0}^{h_0}$ $x_t(\sigma, \psi_0) \in \Omega_\varepsilon^{h_0}$ for all $t \geq \sigma$. Let us fix arbitrary small $\eta \in (0, \varepsilon)$ and choose $\psi_0 \in \mathcal{A}_R^{h_0} \cap \Omega_\eta^{h_0}$ such that $\alpha = v(\sigma, \psi_0) > 0$. Lemma 5 implies that $x_t(\sigma, \psi_0) \in \mathcal{A}_R^{h_0}$ for all $t \geq \sigma + h_0$. It means that $x_t \in \mathcal{A}_R^{h_0} \cap \Omega_\varepsilon^{h_0}$ for $t \geq \sigma + h_0$. By condition 3) of the theorem the functional v is bounded along the given solution $x(\sigma, \psi_0)$ of the system (2).

Denote $t_k = \sigma + kT_1, k = 0, 1, \ldots$. Since $\psi_0 \in \mathcal{A}_R^{h_0}$, then $\|\psi_0\| < R|\psi_0(0)|$. According to condition 1)

$$v(t, x_t(\sigma, \psi)) - v(t_0, x_{t_0}(\sigma, \psi)) \geq \int_{t_0}^{t} \Phi(\tau, x_\tau)\, d\tau.$$

By the same way as in the proof of Theorem 6 we obtain the main inequality

$$v(t_0 + T_1, x_{t_0+T_1}) \geq v(t_0, x_{t_0}) - \delta T_1 |x(t_0)|^d. \tag{26}$$

This inequality is valid for all $(t_0, x_{t_0}) \in [\sigma, \infty) \times (\mathcal{A}_R^{h_0} \cap \Omega_\varepsilon^{h_0})$.

Denote $t_k = \sigma + kT_1, k = 0, 1, \ldots$. Since $\|x_t(\sigma, \psi_0)\| < \varepsilon$ for all $t \geq \sigma$, we obtain from (26) that for any integer $k = 0, 1, 2, \ldots$

$$v(t_k + T_1, x_{t_k+T_1}(\sigma, \psi_0)) \geq v(\sigma, \psi_0) + \delta T_1(|x(t_0)|^d + |x(t_1)|^d + \ldots + |x(t_k)|^d). \tag{27}$$

Note that for any integer k $|x(t_k)| > (1/R)\|x_{t_k}\| \geq (1/R)b^{-1}(v(t_k, x_{t_k})) > (1/R)b^{-1}(\alpha) > 0$. Hence the right-hand side of (27) tends to $+\infty$ as $k \to +\infty$. It contradicts the boundedness of the functional v in the region $[\sigma, \infty) \times (\mathcal{A}_R^{h_0} \cap \Omega_\varepsilon^{h_0})$. The theorem is proved.

5 Remarks

Remark 8. Theorems 6 and 7 are valid also for the systems in the standard form of the type

$$\dot{x} = \mu \mathcal{L}(t, x_t), \tag{28}$$

where μ is a positive small parameter, $\mathcal{L}$ is linear in x_t and $|\mathcal{L}(t, \varphi)| \leq M(t)\|\varphi\|$ for any $t \geq 0$ and $\varphi \in \Omega_H^h$ with some function $M \in \mathcal{UI}(R_+)$.

Remark 9. Condition 3) of Theorem 6 (respectively, condition 5) of Theorem 7) will be fulfilled if there exists the average

$$\{\Phi\}(t_0, x_0) = \lim_{\Delta t \to \infty} \frac{1}{\Delta t} \int_{t_0}^{t_0+\Delta t} \Phi(t, x_0)\, dt \tag{29}$$

and a constant $\delta_0 > 0$ such that for all $t_0 \geq 0$ $\{\Phi\}(t_0, x_0) \leq 2\delta_0 |x_0|^d$ (respectively, $\{\Phi\}(t_0, x_0) \geq 2\delta_0 |x_0|^d$).

6 Examples

By simple examples we present an algorithm for construction of functionals which satisfy all conditions of new theorems on stability given in the previous sections.

Example 10. Consider a nonlinear equation with a time-varying delay

$$\dot{x} = b(t)x^3(\rho(t)), \tag{30}$$

where ρ is a differentiable function, $t - h \leq \rho(t) \leq t$ for all $t > 0$ and some positive constant $h > 0$. Suppose that the function b has a zero average $\{b(t)\}$

and a bounded antiderivative $B, B'(t) = b(t)$, on $t \in [0, \infty)$. To construct an appropriate functional v we take a positive definite function $v_0(x) = x^2/2$. Then

$$\dot{v}_0|_{(30)} = b(t)x(t)x^3(\rho(t)) = \Phi_0(t, x(t), x(\rho(t))).$$

Following to Remark 8, we have to evaluate the average $\{\Phi_0\}(t_0, x_0)$ along the constant solution x_0 of the trivial system. The average $\{\Phi_0(t, x_0, x_0)\} \equiv 0$, since $\{b(t)\} = 0$. Consider a functional

$$v(t, x_t) = v_0(x(t)) + u(t, x(t), x(\rho(t))),$$

where the function $u(t, p, q)$ is a bounded solution of the equation

$$\partial u/\partial t = -\Phi_0(t, p, q) = -b(t)pq^3.$$

Putting $u(t, p, q) = -B(t)pq^3$, we obtain a functional

$$v = v_0 + u = x(t)^2/2 - B(t)x(t)x^3(\rho(t)). \tag{31}$$

Calculating a full derivative of this functional in virtue of the system (30), we get

$$\begin{aligned}\dot{v}|_{(30)} &= \Phi_1(t, x(t), x(\rho(t)), x(\rho(\rho(t)))) \\ &= -B(t)[b(t)x^6(\rho(t)) + 3x(t)x^2(\rho(t))x^3(\rho(\rho(t)))b(\rho(t))\rho'(t)].\end{aligned}$$

A sign of the average $\{\Phi_1(t, x_0, x_0, x_0)\}$ is defined by a sign of the average

$$\delta_0 - \{B(t)(b(t) + 3\rho'(t)b(\rho(t)))\}. \tag{32}$$

According to Theorems 6 and 7 the zero solution of the system (30) is uniformly asymptotically stable if $\delta_0 < 0$ and it is unstable if $\delta_0 > 0$.

Let $b(t) = \cos t, \rho(t) = t - \beta + \alpha \sin \omega t$, where α, β and ω are some constants. Then $B(t) = \sin t, \rho'(t) = 1 + \alpha\omega \cos \omega t$. Substituting given functions to (30), we obtain the equation

$$\dot{x} = \cos t x^3(t - h(t)), \tag{33}$$

where $h(t) = \beta - \alpha \sin \omega t$. The index of stability (32) now has the form

$$\delta_0 = -\{\sin t(1 + \alpha\omega \cos \omega t) \cos(t - \beta + \alpha \sin \omega t)\}, \tag{34}$$

since the average $\{\sin t \cos t\} = 0$. If $\alpha = 0$ the system (33) takes the form

$$\dot{x} = \cos t x^3(t - \beta) \tag{35}$$

with the constant delay β. In this case (34) gives $\delta_0 = -\sin \beta$. Thus the zero solution of the equation (35) is unstable for any $\beta \in (\pi, 2\pi)$, since $\sin \beta < 0$ and all conditions of Theorem 7 are fulfilled.

It is interesting that it is possible to choose the values of the parameters α, β and ω such that the zero solution of the equation (33) with time-varying delay becomes already uniformly asymptotically stable although

$$h(t) = \beta - \alpha \sin \omega t \in (\pi, 2\pi),$$

i.e. for every t the value of the delay $h(t)$ lies in the domain of instability of the trivial solution of the same equation but with a constant delay (35). Indeed, put $\beta = 1.5\pi$, $\alpha = 1.55 < \pi/2$ and $\omega = 2$. Then (34) gives $\delta_0 = -0.04033 < 0$ and all conditions of Theorem 6 on asymptotic stability are fulfilled, but $\sin h(t) < 0$ for all $t \in (-\infty, \infty)$ because $h(t) = 1.5\pi - 1.55 \cos 2t \in [3.16, 6.27] \subset (\pi, 2\pi)$

This phenomenon of changing of the type of stability after replacing of the constant parameter by the continuous function with the same range of values is well-known for ordinary differential equations. It have been first demonstrated for linear equation with deviating argument by A. D. Myshkis [8].

Example 11. [9] Consider the linear equation of Mathieu type with time delay

$$\ddot{x} + \omega^2[x(t) - \mu(2\cos \nu t)x(t-h)] = 0, \tag{36}$$

where μ is small parameter. This equation turns into the well-known Mathieu equation at $h = 0$:

$$\ddot{x} + \omega^2[x(t) - \mu(2\cos \nu t)x(t)] = 0. \tag{37}$$

There are infinite sequence of the so-called regions of dynamical instability for the Mathieu equation (37) at the critical values $\nu = 2\omega/m, m = 1, 2, 3, \ldots$. This phenomenon is called *parametric resonance*. By Theorems 6 and 7 we show that the main resonance $\nu = 2\omega$ also appears in the equation (36) for any delay h. At $\nu \neq 2\omega$ the type of stability depends greatly from h. The delay being introduced may damp the demultiplicative parametric resonances and make the equation either unstable or asymptotically stable.

Introducing complex conjugate variables ζ and $\bar{\zeta}$

$$\zeta \exp(i\omega t) = x - i\frac{\dot{x}}{\omega}, \qquad \bar{\zeta}\exp(-i\omega t) = x + i\frac{\dot{x}}{\omega}, \tag{38}$$

and using more short notations for the variables with deviating argument:

$$\zeta_h = \zeta(t-h), \qquad \zeta_{2h} = \zeta(t-2h), \qquad \ldots,$$

we reduce (36) to the linear system in standard form

$$\dot{\zeta} = \mu Z(t, \zeta_h, \bar{\zeta}_h), \qquad \dot{\bar{\zeta}} = \mu \bar{Z}(t, \zeta_h, \bar{\zeta}_h), \tag{39}$$

where

$$Z(t, \zeta_h, \bar{\zeta}_h) = -0,5i\omega[\zeta_h e^{-i\omega h}(e^{i\nu t} + e^{-i\nu t}) + \bar{\zeta}_h e^{i\omega h}(e^{-i(2\omega+\nu)t} + e^{-i(2\omega-\nu)t})].$$

To construct a suitable functional we start with the positive definite function $v_0(\zeta,\bar\zeta)=\zeta\bar\zeta$. Differentiating it in virtue of (39), we obtain

$$\begin{aligned}\dot v_0|_{(39)} &= \mu\Phi_0(t,\zeta_h,\bar\zeta_h)=\mu 2\mathrm{Re}\left(\frac{\partial v_0}{\partial\zeta}Z\right)\\ &=\mu\mathrm{Re}[(-i\omega)(\bar\zeta\zeta_h e^{-i\omega h}(e^{i\nu t}+e^{-i\nu t})\\ &\quad+\bar\zeta\bar\zeta_h e^{i\omega h}(e^{-i(2\omega+\nu)t}+e^{-i(2\omega-\nu)t})].\end{aligned}\tag{40}$$

It is obvious that average (29) $\{\Phi_0\}=\{\Phi_0(t,\zeta_0,\bar\zeta_0)\}=0$, if $\nu\neq 2\omega$.

Let $\nu=2\omega$, then

$$\{\Phi_0\}=\mathrm{Re}[(-i\omega)\bar\zeta_0^2e^{i\omega h}]=i\omega(\zeta_0^2e^{-i\omega h}-\bar\zeta_0^2e^{i\omega h})$$

Denoting

$$w(\zeta,\bar\zeta)=i(\zeta^2e^{-i\omega h}-\bar\zeta^2e^{i\omega h}),$$

we have that

$$\begin{aligned}\dot w|_{(39)} &= \mu\Psi(t,\zeta_h,\bar\zeta_h)=2\mu\mathrm{Re}\left(\frac{\partial w}{\partial\zeta}Z\right)\\ &=2\mu\mathrm{Re}[\omega(\zeta\zeta_h e^{-i2\omega h}(e^{i2\omega t}+e^{-i2\omega t})+\zeta\bar\zeta_h(e^{-i4\omega t}+1))].\end{aligned}$$

The average $\{\Psi(t,\zeta_0,\bar\zeta_0)\}=2\omega|\zeta_0|^2$ is positive definite and for small enough $|\mu|$ all conditions of Theorem 7 on instability are fulfilled, moreover, $\{\Psi\}$ does not depend on h.

Suppose now that $\nu\neq 2\omega$, then $\{\Phi_0\}\equiv 0$. Following to the theory of the generalized Lyapunov functions [5] we have to construct the so-called *perturbed functional*

$$V_1=v_0+\mu v_1,\tag{41}$$

where a *perturbation* v_1 of the functional (function) v_0 is calculated by a bounded solution of the equation

$$\partial v_1/\partial t=-\Phi_0.$$

Since

$$\begin{aligned}-\int\left(\frac{\partial v_0}{\partial\zeta}Z\right)dt &= (\omega/2)\Bigg[\bar\zeta\zeta_h e^{-i\omega h}\left(\frac{e^{i\nu t}-e^{-i\nu t}}{\nu}\right)\\ &\quad-\bar\zeta\bar\zeta_h e^{i\omega h}\left(\frac{e^{-i(2\omega+\nu)t}}{2\omega+\nu}+\frac{e^{-i(2\omega-\nu)t}}{2\omega-\nu}\right)\Bigg]+\mathrm{const},\end{aligned}$$

we can take the functional v_1 in the form

$$\begin{aligned}v_1(t,\zeta,\bar\zeta,\zeta_h,\bar\zeta_h) &= \omega\mathrm{Re}\Bigg[\bar\zeta\zeta_h e^{-i\omega h}\left(\frac{e^{i\nu t}-e^{-i\nu t}}{\nu}\right)\\ &\quad-\bar\zeta\bar\zeta_h e^{i\omega h}\left(\frac{e^{-i(2\omega+\nu)t}}{2\omega+\nu}+\frac{e^{-i(2\omega-\nu)t}}{2\omega-\nu}\right)\Bigg].\end{aligned}$$

Then

$$\dot{V}_1\Big|_{(39)} = \dot{v}_0|_{(39)} + \mu\frac{\partial v_1}{\partial t} + \mu^2 2\mathrm{Re}\left(\frac{\partial v_1}{\partial \zeta}Z + \frac{\partial v_1}{\partial \zeta_h}Z_h\right) = \mu^2 \Phi_1.$$

Making necessary calculations, we obtain

$$\begin{aligned}
&\frac{\partial v_1}{\partial \zeta}Z + \frac{\partial v_1}{\partial \zeta_h}Z_h = \\
&\quad = (-i\omega^2/4)\Big[\bar{\zeta}_h\zeta_h\Big((e^{-i2\nu t} - e^{i2\nu t})/\nu - \frac{4\omega}{4\omega^2-\nu^2} - \frac{e^{i2\nu t}}{2\omega+\nu} - \frac{e^{-i2\nu t}}{2\omega-\nu}\Big) \\
&\qquad + \bar{\zeta}_h^2 e^{i2\omega h}(e^{-i(2\omega+2\nu)t} - e^{-i(2\omega-2\nu)t})/\nu \\
&\qquad - \zeta_h^2 e^{-i2\omega h}\left(\frac{e^{i(2\omega+2\nu)t}+e^{i2\omega t}}{2\omega+\nu} + \frac{e^{i(2\omega-2\nu)t}+e^{i2\omega t}}{2\omega-\nu}\right) \\
&\qquad + \bar{\zeta}\zeta_{2h}e^{-i2\omega h}(e^{-i2\nu t}e^{-i\nu h} - e^{-i\nu h} + e^{i\nu h} - e^{-i2\nu t}e^{i\nu h})/\nu \\
&\qquad + \bar{\zeta}\bar{\zeta}_{2h}(1/\nu)(e^{-i2\omega t}e^{i(2\omega+\nu)h} + e^{-i(2\omega-2\nu)t}e^{i(2\omega-\nu)h} \\
&\qquad - e^{-i(2\omega+2\nu)t}e^{i(2\omega+\nu)h} - e^{-i2\omega t}e^{i(2\omega-\nu)h}) \\
&\qquad - \zeta\zeta_{2h}e^{-i2\omega h}\big((e^{-i\nu h}e^{i(2\omega+2\nu)t} + e^{i2\omega t}e^{i\nu h})/(2\omega+\nu) \\
&\qquad + (e^{i(2\omega-2\nu)t}e^{i\nu h} + e^{-i\nu h}e^{i2\omega t})/(2\omega-\nu)\big) \\
&\qquad - \zeta\bar{\zeta}_{2h}\big((e^{i(2\omega+\nu)h} + e^{i2\nu t}e^{i(2\omega-\nu)h})/(2\omega+\nu) \\
&\qquad\qquad + (e^{i(2\omega+\nu)h}e^{i2\nu t} + e^{i(2\omega-\nu)h})/(2\omega-\nu)\big)\Big].
\end{aligned}$$

Here we use a notation: $Z_h = Z(t-h, \zeta_{2h}, \bar{\zeta}_{2h})$. Consequently, the average of the functional $\Phi_1(t, \zeta_h, \bar{\zeta}_h, \zeta_{2h}, \bar{\zeta}_{2h})$ for $\nu \neq 2\omega, \nu \neq \omega$ and $\zeta = \zeta_0$ has the form

$$\{\Phi_1\} = \frac{\omega^3}{2\nu}|\zeta_0|^2\left(\frac{\sin(2\omega+\nu)h}{2\omega+\nu} - \frac{\sin(2\omega-\nu)h}{2\omega-\nu}\right).$$

It is not equal to zero for all values of h, in spite of the zeros of a function

$$\sigma_\nu(\omega h) = \frac{\sin(2+\nu/\omega)\omega h}{2+\nu/\omega} - \frac{\sin(2-\nu/\omega)\omega h}{2-\nu/\omega}.$$

Since the functional (41) is positively defined in the cone $\mathcal{A}_R \subset C_h$. $R > 1$, for small enough μ, the equation (36) is uniformly asymptotically stable for $\sigma_\nu(\omega h) < 0$ and it is unstable for $\sigma_\nu(\omega h) > 0$.

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