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In: Zuzana Došlá and Jaromír Kuben and Jaromír Vosmanský (eds.): Proceedings of Equadiff 9, Conference on Differential Equations and Their Applications, Brno, August 25-29, 1997, [Part 3] Papers. Masaryk University, Brno, 1998. CD-ROM. pp. 229--235.

Persistent URL: http://dml.cz/dmlcz/700292

EQUADIFF 9 CD ROM, Brno 1997 PAPERS
Masaryk University pp. 229–235

On the Limit Cycle of the van der Pol Equation

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Abstract. In the paper, we estimate the amplitude (maximal x-value) of the limit cycle of the van der Pol equation

$$\dot{x} = y - \mu(x^3/3 - x), \quad \dot{y} = -x$$

from above by $\rho(\mu) < 2.3439$ for every $\mu \neq 0$. The result is an improvement of the author's previous estimation $\rho(\mu) < 2.5425$.

AMS Subject Classification. 34C05, 58F21

Keywords. Van der Pol equation, limit cycle, amplitude

1 Introduction

We are interested in the limit cycle (isolated periodic orbit) of the Liénard equation:

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x). \tag{L}$$

The following is our result.

Theorem A. *Suppose that Liénard equation satisfies the following conditions:* (1) F, g *are of class* C^1 *and odd;* (2) $g(x)$ *has the same sign as* x*;* (3) F *has a positive zero* β *such that* $F(x) < 0$ *on* $(0, \beta)$ *and* > 0 *on* (β, ∞)*;* (4) *there are two piecewise differentiable, continuous mappings* $\phi, \psi : [0, \beta] \to [\beta, \infty)$ *such that*
(i) $-\phi'(x)g(\phi(x))F(\phi(x)) \geq -g(x)F(x)$, (ii) $-\phi'(x)f(\phi(x)) \geq -f(x)$,
(iii) $\psi'(x)g(\psi(x))F(\psi(x)) \geq -g(x)F(x)$, (iv) $\psi'(x)f(\psi(x)) \geq f(x)$,
(v) $\psi'(x)g(\psi(x)) \leq g(x)$, (vi) $\phi(0) \leq \psi(\beta)$,
where $f = F'$*. Then it has a periodic orbit in the strip* $|x| < \psi(\beta)$*.*

The above theorem is effective to estimate the amplitude (maximal x-value) of the limit cycle of the van der Pol equation:

$$\dot{x} = y - \mu(x^3/3 - x), \quad \dot{y} = -x. \tag{vdP}$$

We know that the van der Pol equation has a unique limit cycle for every $\mu \neq 0$; see [O] for example. The following is an application of Theorem A.

This is the preliminary version of the paper.

Theorem B. *The amplitude $\rho(\mu)$ of the limit cycle of the van der Pol equation is estimated by $\rho(\mu) < 2.3439$ for every $\mu \neq 0$.*

The upper bound 2.3439 is better than previous results, namely, 2.8025 of Alsholm [A] and 2.5425 of the author [O]. Due to a computer experiment, we expect that the amplitude $\rho(\mu) < 2.0235$ for every $\mu \neq 0$. So Theorem B is not a sharp result in comparison with it. We give the result of the experiment in Section 4.

2 Proof of Theorem A

We consider an orbit γ which starts from a point on the left half of the curve $y = F(x)$ and reaches to the right half of it. Then we can regard the y-coordinate of γ as a function of x, that is, $y = y(x)$. In the proof of Theorem A, we use the following notation:

$$v_1(x) = y(x) - F(x), \quad v_2(x) = y(-x) + F(x). \tag{1}$$

Then the functions v_1, v_2 must satisfy the following differential equations:

$$\frac{dv_1}{dx} = -\frac{g(x)}{v_1} - f(x), \quad \frac{dv_2}{dx} = -\frac{g(x)}{v_2} + f(x). \tag{2}$$

By the definition of γ, we know that $v_1(x), v_2(x) \geq 0$ on $[0, \psi(\beta)]$.

Proof (of Theorem A). We assume that the orbit γ starts from the curve $y = F(x)$ at $x = -\psi(\beta)$, that is, $v_2(\psi(\beta)) = 0$. We want to prove that the orbit γ gets across the curve at the left-hand side of $x = \psi(\beta)$. To prove it by a contradiction, we assume that $v_1(x)$ is defined on $[0, \psi(\beta)]$.

By using (i), we know that $\phi'(x) < 0$ on $(0, \beta)$. So by using (ii), we calculate as follows:

$$\begin{aligned}\frac{d}{dx}\Big(v_1(x) - v_1(\phi(x))\Big) \\ = -\frac{g(x)}{v_1(x)} + \frac{\phi'(x)g(\phi(x))}{v_1(\phi(x))} - f(x) + \phi'(x)f(\phi(x)) \leq 0.\end{aligned} \tag{3}$$

By integrating it on $[x, \beta]$, we obtain that

$$v_1(x) - v_1(\phi(x)) \geq v_1(\beta) - v_1(\phi(\beta)) = y(\beta) - y(\phi(\beta)) + F(\phi(\beta)) > 0 \tag{4}$$

because $y(x)$ is strictly decreasing on $[-\phi(\beta), \phi(\beta)]$.

On the other hand, by using (iv), (v), we calculate as follows:

$$\begin{aligned}\frac{d}{dx}\Big(v_2(x) - v_2(\psi(x))\Big) \\ = -\frac{g(x)}{v_2(x)} + \frac{\psi'(x)g(\psi(x))}{v_2(\psi(x))} + f(x) - \psi'(x)f(\psi(x)) \\ \leq \frac{g(x)}{v_2(x)v_2(\psi(x))}\Big(v_2(x) - v_2(\psi(x))\Big).\end{aligned} \tag{5}$$

By integrating it on $[x, \beta]$, we obtain that

$$v_2(x) - v_2(\psi(x)) \geq \Big(v_2(\beta) - v_2(\psi(\beta))\Big) \exp\Big(-\int_x^\beta \frac{g(u)du}{v_2(u)v_2(\psi(u))}\Big) > 0. \quad (6)$$

We can easily confirm the following equality:

$$\frac{d}{dx}\Big(\frac{1}{2}y(x)^2 + \int_0^x g(u)du\Big) = -\frac{g(x)F(x)}{y(x) - F(x)}. \quad (7)$$

By integrating it on $[0, \psi(\beta)]$, we obtain that

$$\frac{1}{2}\Big(y(\psi(\beta))^2 - y(-\psi(\beta))^2\Big) = -\int_0^{\psi(\beta)} \frac{g(x)F(x)}{v_1(x)}dx - \int_0^{\psi(\beta)} \frac{g(x)F(x)}{v_2(x)}dx. \quad (8)$$

By using (i), (4), we calculate the first term of (8) as follows:

$$\begin{aligned} &\leq -\int_0^\beta \frac{g(x)F(x)}{v_1(x)}dx - \int_{\phi(\beta)}^{\phi(0)} \frac{g(x)F(x)}{v_1(x)}dx \\ &\qquad = -\int_0^\beta \frac{g(x)F(x)}{v_1(x)}dx + \int_0^\beta \frac{\phi'(x)g(\phi(x))F(\phi(x))}{v_1(\phi(x))}dx < 0. \quad (9) \end{aligned}$$

On the other hand, by using (iii), (6), we calculate the second term of (8) as follows:

$$\begin{aligned} &\leq -\int_0^\beta \frac{g(x)F(x)}{v_2(x)}dx - \int_{\psi(0)}^{\psi(\beta)} \frac{g(x)F(x)}{v_2(x)}dx \\ &\qquad = -\int_0^\beta \frac{g(x)F(x)}{v_2(x)}dx - \int_0^\beta \frac{\psi'(x)g(\psi(x))F(\psi(x))}{v_2(\psi(x))}dx < 0. \quad (10) \end{aligned}$$

By combining (8), (9), (10), we obtain that

$$y(\psi(\beta))^2 < y(-\psi(\beta))^2 = F(\psi(\beta))^2. \quad (11)$$

It is in contradiction with $v_1(\psi(\beta)) \geq 0$. So the function $v_1(x)$ does not defined on $[0, \psi(\beta)]$, that is, the orbit γ gets across the curve $y = F(x)$ at the left-hand side of $x = \psi(\beta)$. Thus the orbit γ winds toward inside. On the other hand, every orbit near the origin winds toward outside. Hence the equation has a periodic orbit in the strip $|x| < \psi(\beta)$. □

3 Proof of Theorem B

In the proof of Theorem B, we use the following functions:

$$P(x) := \frac{f(x)}{g(x)} = \mu\Big(x - \frac{1}{x}\Big), \quad Q(x) := \frac{f(x)}{g(x)F(x)} = \frac{3(x^2-1)}{x^4 - 3x^2}. \quad (12)$$

By checking the derivatives, we know that the function P is strictly increasing on $(0, \infty)$ and that the function Q is strictly decreasing on $(0, \sqrt{3})$ and on $(\sqrt{3}, \infty)$.

Proof (of Theorem B). We can assume without loss of generality that $\mu > 0$ because the transformation $(x, y, t, \mu) \to (x, -y, -t, -\mu)$ preserves the form of the equation. We first define $\phi(x)$ by the following algebraic equation:

$$\int_x^\phi uF(u)du = \frac{\mu}{15}(\phi^5 - 5\phi^3 - x^5 + 5x^3) = 0. \tag{13}$$

Of course, $\phi(\sqrt{3}) = \sqrt{3}$. By differentiating it, we obtain that

$$-\phi'(x)\phi(x)F(\phi(x)) + xF(x) = 0. \tag{14}$$

Since $\phi'(x) < 0$ on $[0, \sqrt{3}]$, the mapping ϕ is strictly decreasing (orientation reversing) on it. Since the function $-Q(\phi(x)) + Q(x)$ is strictly decreasing on $(0, \sqrt{3})$, it has a unique zero ξ_1 in $(0, \sqrt{3})$. A computer experiment indicates that $\xi_1 \approx 0.6941$, $\xi_2 := \phi(\xi_1) \approx 2.2043$. By substituting $\phi'(x)$ from (14) and by the definition of ξ_1, we obtain that

$$\phi'(x)f(\phi(x)) - f(x) = -xF(x)\Big(-Q(\phi(x)) + Q(x)\Big) \leq 0 \tag{15}$$

on $[\xi_1, \sqrt{3}]$. Since (15) does not hold on $[0, \xi_1)$, the definition (13) is valid only on $[\xi_1, \sqrt{3}]$.

On the interval $[0, \xi_1)$, we define $\phi(x)$ by the following algebraic equation:

$$\int_{\xi_2}^\phi f(u)du + \int_x^{\xi_1} f(u)du = \frac{\mu}{3}(\phi^3 - 3\phi - x^3 + 3x - \xi_2^3 + 3\xi_2 + \xi_1^3 - 3\xi_1) = 0. \tag{16}$$

By differentiating it, we obtain that

$$\phi'(x)f(\phi(x)) - f(x) = 0 \tag{17}$$

on $[0, \xi_1)$. By substituting $\phi'(x)$ from (17) and by the definition of ξ_1, we obtain that

$$-\phi'(x)\phi(x)F(\phi(x)) + xF(x) = -\frac{xF(x)}{Q(\phi(x))}\Big(-Q(\phi(x)) + Q(x)\Big) \geq 0 \tag{18}$$

on $[0, \xi_1)$. Hence the mapping ϕ satisfies (i), (ii) of Theorem A.

We first define $\psi(x)$ by the following algebraic equation:

$$\int_{\theta_2}^\psi uF(u)du + \int_{\theta_1}^x uF(u)du = \frac{\mu}{15}(\psi^5 - 5\psi^3 + x^5 - 5x^3 + 4\sqrt{6}) = 0, \tag{19}$$

where $\theta_1, \theta_2 := \sqrt{2 \mp \sqrt{3}} = (\sqrt{3} \mp 1)/\sqrt{2}$. Of course, $\psi(\theta_1) = \theta_2$. By differentiating it, we obtain that

$$\psi'(x)\psi(x)F(\psi(x)) + xF(x) = 0. \tag{20}$$

Since $\psi'(x) > 0$ on $[0, \sqrt{3}]$, the mapping ψ is strictly increasing (orientation preserving) on it. Since the function $Q(\psi(x)) + Q(x)$ is strictly decreasing on $(0, \sqrt{3})$, it has a unique zero η_1 in $(0, \sqrt{3})$. A computer experiment indicates that $\eta_1 \approx 1.3784$, $\eta_2 := \psi(\eta_1) \approx 2.2006$. By substituting $\psi'(x)$ from (20) and by the definition of η_1, we obtain that

$$\psi'(x)f(\psi(x)) - f(x) = -xF(x)\Big(Q(\psi(x)) + Q(x)\Big) \geq 0 \tag{21}$$

on $[0, \eta_1]$. Since (21) does not hold on $(\eta_1, \sqrt{3}]$, the definition (19) is valid only on $[0, \eta_1]$.

On the interval $(\eta_1, \sqrt{3}]$, we define $\psi(x)$ by the following algebraic equation:

$$\int_{\eta_2}^{\psi} f(u)du - \int_{\eta_1}^{x} f(u)du = \frac{\mu}{3}(\psi^3 - 3\psi - x^3 + 3x - \eta_2^3 + 3\eta_2 + \eta_1^3 - 3\eta_1) = 0. \tag{22}$$

By differentiating it, we obtain that

$$\psi'(x)f(\psi(x)) - f(x) = 0 \tag{23}$$

on $(\eta_1, \sqrt{3}]$. By substituting $\psi'(x)$ from (23) and by the definition of η_1, we obtain that

$$\psi'(x)\psi(x)F(\psi(x)) + xF(x) = \frac{xF(x)}{Q(\psi(x))}\Big(Q(\psi(x)) + Q(x)\Big) \geq 0 \tag{24}$$

on $(\eta_1, \sqrt{3}]$. Hence the mapping ψ satisfies (iii), (iv) of Theorem A.

To prove (v), we prepare the mapping $\chi(x) := \sqrt{x^2 + 2\sqrt{3}}$. By the proof of Example 2 of [O], we obtain that

$$F(\chi(x)) \geq -F(x) \tag{25}$$

on $[0, \sqrt{3}]$. By combining (20) and (25), we obtain that

$$\chi'(x)\chi(x)F(\chi(x)) \geq -xF(x) = \psi'(x)\psi(x)F(\psi(x)). \tag{26}$$

By integrating it on $[x, \theta_1]$, we obtain that

$$\int_{\chi(x)}^{\psi(x)} uF(u)du \geq 0 \tag{27}$$

on $[0, \theta_1]$. Since $uF(u) > 0$ on $(\sqrt{3}, \infty)$, we obtain that $\psi(x) \geq \chi(x)$ on $[0, \theta_1]$. So we obtain that

$$F(\psi(x)) \geq F(\chi(x)) \geq -F(x) \quad \text{on } [0, \theta_1]. \tag{28}$$

To prove the same inequality as (28) on $(\theta_1, \eta_1]$, we consider the minimum of the function $F(\psi) + F(x)$ under the restriction (19). We denote by ψ_0, x_0 the variables which attain the minimum. To find the minimum, we consider the following function:

$$\Lambda(\psi, x) = F(\psi) + F(x) - \lambda\Big(\int_{\theta_2}^{\psi} uF(u)du + \int_{\theta_1}^{x} uF(u)du\Big). \tag{29}$$

By the Lagrange's method of indeterminate coefficients, we obtain that

$$\Lambda_\psi(\psi_0, x_0) = f(\psi_0) - \lambda\psi_0 F(\psi_0) = 0, \tag{30}$$

$$\Lambda_x(\psi_0, x_0) = f(x_0) - \lambda x_0 F(x_0) = 0. \tag{31}$$

By the first equality, we obtain that $\lambda > 0$. So we obtain that

$$\begin{aligned} F(\psi(x)) + F(x) \geq F(\psi_0) + F(x_0) &= (1/\lambda)\Big(P(\psi_0) + P(x_0)\Big) \\ &\geq (1/\lambda)\Big(P(\theta_2) + P(\theta_1)\Big) = 0 \end{aligned} \tag{32}$$

on $(\theta_1, \eta_1]$. By substituting $\psi'(x)$ from (20) and by using (28) and (32), we obtain that

$$x - \psi'(x)\psi(x) = \frac{x}{F(\psi(x))}\Big(F(\psi(x)) + F(x)\Big) \geq 0 \tag{33}$$

on $[0, \eta_1]$. On the other hand, by substituting $\psi'(x)$ from (23), we obtain that

$$x - \psi'(x)\psi(x) = \frac{x}{P(\psi(x))}\Big(P(\psi(x)) - P(x)\Big) \geq 0 \tag{34}$$

on $(\eta_1, \sqrt{3}]$. Hence the mappings ϕ, ψ satisfy all the conditions of Theorem A except (vi).

A computer experiment indicates that $\phi(0) \approx 2.3439$, $\psi(\sqrt{3}) \approx 2.3233$. So we must replace ψ by the following mapping:

$$\hat{\psi}(x) := \sqrt{\psi(x)^2 - \psi(\beta)^2 + \phi(0)^2}\,. \tag{35}$$

Of course, $\hat{\psi}(\beta) = \phi(0)$. Moreover, we can calculate as follows:

$$\hat{\psi}'(x)\hat{\psi}(x) = \psi'(x)\psi(x) \leq x, \tag{36}$$

$$\begin{aligned} \hat{\psi}'(x)\hat{\psi}(x)F(\hat{\psi}(x)) &= \psi'(x)\psi(x)F(\hat{\psi}(x)) \\ &\geq \psi'(x)\psi(x)F(\psi(x)) \geq -xF(x), \end{aligned} \tag{37}$$

$$\begin{aligned} \hat{\psi}'(x)f(\hat{\psi}(x)) &= \psi'(x)\psi(x)P(\hat{\psi}(x)) \geq \psi'(x)\psi(x)P(\psi(x)) \\ &= \psi'(x)f(\psi(x)) \geq f(x). \end{aligned} \tag{38}$$

Hence the mappings $\phi, \hat{\psi}$ satisfy all the conditions of Theorem A. □

4 A Conjecture

Since the limit cycle of the van der Pol equation is unique, its amplitude $\rho(\mu)$ is a continuous function of the parameter $\mu \neq 0$. In [L], the following facts are proved:

$$\rho(\mu) \to 2 \text{ as } \mu \to 0, \quad \rho(\mu) \to 2 \text{ as } \mu \to \infty. \tag{39}$$

More precisely, it is proved in [H] that $\rho(\mu) = 2+(7/96)\mu^2+O(\mu^3)$ for sufficiently small $\mu > 0$ and in [C] that $\rho(\mu) = 2+(0.7793\cdots)\mu^{-4/3}+o(\mu^{-4/3})$ for sufficiently large $\mu > 0$.

By a computer experiment, we have the following table.

μ	$\downarrow 0$	0.1	1.0	2.0	3.0	3.2
ρ	$\downarrow 2$	2.00010	2.00862	2.01989	2.02330	2.02341
μ	3.3	3.4	4.0	5.0	10	$\uparrow \infty$
ρ	2.02342	2.02341	2.02296	2.02151	2.01429	$\downarrow 2$

We calculate the amplitude ρ of the above table by using the Runge-Kutta method with a step size 2^{-20}. In comparison with the above table, we realize that Theorem B is not a sharp result. So we want to pose the following conjecture.

Conjecture. The amplitude $\rho(\mu)$ of the limit cycle of the van der Pol equation is estimated by $2 < \rho(\mu) < 2.0235$ for every $\mu \neq 0$.

However, to estimate the amplitude is a very difficult problem. An attempt to estimate the amplitude is done by Giacomini and Neukirch [GN].

Acknowledgement. The author wishes to thank Professors K. Shiraiwa and K. Yamato for reading the manuscript.

References

[A] P. Alsholm, Existence of limit cycles for generalized Liénard equation, J. Math. Anal. Appl. 171 (1992), 242–255.

[C] M. L. Cartwright, Van der Pol's equation for relaxation oscillation, in "Contributions to the Theory of Non-linear Oscillations II", S. Lefschetz, ed., Ann. of Math. Studies, vol. 29, Princeton Univ. Press, 1952, pp. 3–18.

[GN] H. Giacomini and S. Neukirch, On the number of limit cycles of Liénard equation, Physical Review E, to appear.

[H] W. T. van Horssen, A perturbation method based on integrating factors, SIAM J. Appl. Math., to appear.

[L] S. Lefschetz, "Differential Equations: Geometric Theory", 2nd Ed., Interscience, 1963; reprint, Dover, New York, 1977.

[O] K. Odani, Existence of exactly N periodic solutions for Liénard systems, Funkcialaj Ekvacioj **39** (1996), 217–234.

[Y] Y.-Q. Ye et al., "Theory of Limit Cycles", Transl. of Math. Monographs, vol. 66, Amer. Math. Soc., 1986. (Eng. transl.)

[Z] Z.-F. Zhang et al., "Qualitative Theory of Differential Equations", Transl. of Math. Monographs, vol. 102, Amer. Math. Soc., 1992. (Eng. transl.)