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Some Global Bifurcation Problems for Variational Inequalities

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Abstract. The paper presents several examples of bifurcation problems for variational inequalities and discusses an abstract framework for treating such problems. This abstract framework is applied to analyze some of the problems stated.

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1 Introduction

This paper is based on a lecture presented by the second author at *Equadiff 9* held during the last week of August, 1997 in Brno, Czech Republic. The purpose of the lecture was to present several illustrations of global bifurcation phenomena in variational inequalities and to present some general framework for the analysis of such problems. Thus we present and discuss several examples and show how the global bifurcation results derived in [9] may be applied.

We first present examples of bifurcation problems which may be formulated as variational inequalities, then provide an abstract setting for these problems and state and sketch a proof of a global bifurcation theorem which will apply in these situations and finally provide a (partial) bifurcation analysis for the examples given.

When studying buckling phenomena of constrained elastic systems, one is led in a very natural way to bifurcation problems for variational inequalities. For example, the problem of the buckling of a slender column (beam) that is constrained by some obstacles leads to a problem for variational inequalities, simply because one searches for extremal points of an energy functional in a space of possible displacements determined by the obstacles, and hence these extremal points, which in the absence of constraints result in the Euler-Lagrange differential equations, now will be characterized as solutions of inequalities.

2 Some examples

In this section we present several examples of bifurcation problems which may be formulated as bifurcation problems for variational inequalities.

2.1 A unilateral problem

Consider the following ordinary differential equation

$$-u'' + u = \lambda(u + u^3),\ \ t \in (0, \pi), \tag{2.1}$$

subject to the unilateral constraints

$$\begin{cases} 0 \le u(0),\ \ 0 \le u(\pi) \\ u'(0) \le 0 \le u'(\pi) \\ u(0)u'(0) = 0 = u(\pi)u'(\pi). \end{cases} \tag{2.2}$$

Since nontrivial solutions of (2.1) may not have multiple zeros, we see that the above problem includes four different types of boundary value problems, namely problems subject to the following conditions:

1. *Dirichlet boundary conditions:*

$$u(0) = 0 = u(\pi), \tag{2.3}$$

where, however λ must be restricted so that the second of the unilateral conditions (2.2) hold, i.e.

$$u'(0) < 0 < u'(\pi). \tag{2.4}$$

Thus, for example, the problem may not have any solutions u, with $u(t) > 0$, $t \in (0, \pi)$, nor any solutions u with $u(t) > 0$ for t in a neighborhood of 0 and $u(t) < 0$ for t in a neighborhood of π. Thus, imitating the bifurcation analysis for nonlinear Sturm-Liouville problems, we would surmise that the values

$$\lambda = n^2 + 1,\ \ n = 1, 3, \cdots \tag{2.5}$$

are bifurcation points, whereas the values

$$\lambda = n^2 + 1,\ \ n = 2, 4, \cdots \tag{2.6}$$

are not. Furthermore, changing the sign of a solution will no longer yield a solution. Solutions must have an even number of zeros interior to $(0, \pi)$.

2. *Neumann boundary conditions:*

$$u'(0) = 0 = u'(\pi), \tag{2.7}$$

where, however λ must be restricted so that the first of the unilateral conditions (2.2) hold, i.e.

$$u(0), u(\pi) > 0. \tag{2.8}$$

Again, using the bifurcation analysis for nonlinear Sturm-Liouville problems, we find that the values

$$\lambda = n^2 + 1,\ \ n = 0, 2, 4, \cdots \tag{2.9}$$

are bifurcation points, whereas the values

$$\lambda = n^2 + 1,\ \ n = 1, 3, \cdots \tag{2.10}$$

are not. Again, changing the sign of a solution will no longer yield a solution and solutions must have an even number of zeros interior to $(0, \pi)$.

3. *Mixed Dirichlet and Neumann boundary conditions:*

$$u(0) = 0 = u'(\pi), \tag{2.11}$$

where, however λ must be restricted so that the first and the second of the unilateral conditions (2.2) hold, i.e.

$$u'(0) < 0,\ \ u(\pi) > 0. \tag{2.12}$$

As above, we compute that the values

$$\lambda = \left(\frac{2n-1}{2}\right)^2 + 1,\ \ n = 1, 3, 5, \cdots \tag{2.13}$$

are bifurcation points, whereas the values

$$\lambda = \left(\frac{2n-1}{2}\right)^2 + 1,\ \ n = 2, 4, 6, \cdots \tag{2.14}$$

are not. Changing the sign of a solution will no longer yield a solution and these solutions must have an odd number of simple zeros interior to $(0, \pi)$.

4. *Mixed Neumann and Dirichlet boundary conditions:*

$$u'(0) = 0 = u(\pi), \tag{2.15}$$

where, however λ must be restricted so that the first and the second of the unilateral conditions (2.2) hold, i.e.

$$u(0) > 0,\ \ u'(\pi) > 0. \tag{2.16}$$

In this case we obtain the set of bifurcation points as for the other set of mixed boundary conditions considered above.

Let us formulate the above problem as an equivalent bifurcation problem for a variational inequality. To this end we consider the Sobolev space $H^1(0,\pi)$ (of all $L^2(0,\pi)$ functions with a square integrable first distributional derivative) and let the (closed and convex) set K be defined by

$$K = \{u \in H^1(0,\pi) : u(0) \geq 0,\ u(\pi) \geq 0\}.$$

Then, if u solves (2.2), u will be in $H^2(0,\pi)$ and hence $u \in C^1[0,\pi]$. Therefore if u also satisfies the boundary constraints (2.3), we may multiply (2.1) by an arbitrary $v \in K$ and an integration by parts and the boundary constraints yield

$$\begin{cases} \displaystyle\int_0^\pi u'(v-u)' + u(v-u) - \lambda(u+u^3)(v-u) \geq 0,\ \forall v \in K, \\ u \in K, \end{cases} \tag{2.17}$$

which is a variational inequality. Conversely, if u solves the variational inequality (2.17), using the density of $C_0^\infty(0,\pi)$ in K, we easily conclude that u actually solves (2.1), (2.2).

If we denote by I_K, the indicator function of the set K, i.e.

$$I_K(u) = \begin{cases} 0, & u \in K \\ \infty, & u \notin K, \end{cases}$$

then we see that the variational inequality (2.17) is equivalent to the variational inequality

$$\begin{cases} \displaystyle\int_0^\pi u'(v-u)' + u(v-u) - \lambda(u+u^3)(v-u) + I_K(v) - I_K(u) \geq 0 \\ \forall v \in H^1(0,\pi) \\ u \in H^1(0,\pi). \end{cases} \tag{2.18}$$

We note here, that because of the convexity and closedness of K, the functional I_K is a lower semicontinuous convex functional on the (Hilbert) space $H^1(0,\pi)$.

2.2 A unilateral problem for a semilinear elliptic equation

A higher dimensional analogue of the problem discussed above in section 2.1 is the following unilateral problem. Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 2$, and consider the semilinear elliptic equation

$$-\Delta u + u = \lambda(u + g(u)),\ x \in \Omega, \tag{2.19}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a smooth odd function with $g'(0) = 0$ and $|g(u)| \leq a + b|u|^s,\ 1 \leq s < \frac{N+2}{N-2}$. Let the following unilateral constraints be imposed

$$\begin{cases} u(x) \geq 0,\ \frac{\partial u}{\partial \nu} \geq 0,\ x \in \partial\Omega \\ u(x)\frac{\partial u}{\partial \nu} = 0,\ x \in \partial\Omega, \end{cases} \tag{2.20}$$

where ν is the unit normal vector field to $\partial\Omega$.

In this case, if we consider the Sobolev space $H^1(\Omega)$ and let

$$K = \{u \in H^1(\Omega) : u(x) \geq 0,\ x \in \partial\Omega \text{ (in the sense of traces)}\},$$

then the above unilateral problem is equivalent to the variational inequality

$$\begin{cases} \int_\Omega \nabla u \nabla(v-u) + u(v-u) - \lambda(u+g(u))(v-u) + I_K(v) - I_K(u) \geq 0, \\ \forall v \in H^1(\Omega), \\ u \in H^1(\Omega). \end{cases} \tag{2.21}$$

It is again apparent that the special Dirichlet problem, i.e. equation (2.19) subject to the boundary condition

$$u = 0,\ x \in \partial\Omega,$$

and the Neumann problem, i.e. equation (2.19) subject to

$$\frac{\partial u}{\partial \nu} = 0,\ x \in \partial\Omega$$

will yield some of the bifurcation points for problem (2.21). However, one very quickly sees that much more is needed to detect other bifurcation points.

2.3 A simply supported, or clamped, slender beam subject to elastic obstacles

In this example, we consider a bifurcation problem for a beam resting between two foundations (one above and one below, with partial contact along its length) with nonlinear elastic laws. This problem can be modeled by the following variational inequality:

$$\begin{cases} \int_0^a u''(v-u)'' - \lambda \int_0^a \frac{u'}{\sqrt{1+u'^2}}(v-u)' \\ + \left[\int_{I_1} k_1(v^-)^\gamma + \int_{I_2} k_2(v^+)^\beta\right] \\ - \left[\int_{I_1} k_1(u^-)^\gamma + \int_{I_2} k_2(u^+)^\beta\right] \ \geq\ 0,\ \forall v \in E, \\ u \in E. \end{cases} \tag{2.22}$$

Here, $[0,a]$ $(a > 0)$ is the interval occupied by the beam, and $E = H_0^2(0,a)$, or $E = H^2(0,a) \cap H_0^1(0,a)$ depending on whether the beam is clamped or is simply supported at the ends 0 and a. $I_1, I_2 \subset (0,a), |I_1|, |I_2| > 0$ are closed, disjoint sets representing the domain of possible contact between the beam and the foundations.

We refer to [12], [13], and [14], for the physical motivation in deriving such a model.

Because $u \mapsto u^+, u^-,\ u \in \mathbb{R}$ are nonnegative and convex, we see that the functional j, given by

$$j(u) = \int_{I_1} k_1(u^-)^\gamma + \int_{I_2} k_2(u^+)^\beta,$$

is well defined, with values in $[0, \infty]$. Moreover, j is convex and nonnegative, and $j(0) = 0$. Using Fatou's lemma, we find that j is lower semicontinuous on V.

2.4 Bifurcation problems for Navier-Stokes flows

We consider here bifurcation problems for some (nonlinear) variational inequalities associated with the Navier-Stokes equation, subject to different types of unilateral constraints (cf. [10]). Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary. We are concerned with variational inequalities of the form:

$$\begin{cases} \nu \displaystyle\int_\Omega Du : D(v-u) + b(u,u,v-u) + j(v) - j(u) \\ \geq \displaystyle\int_\Omega g(x,u,\lambda)\cdot(v-u),\ \forall v \in E \\ u \in E. \end{cases} \tag{2.23}$$

Here $E = \{v \in [H_0^1(\Omega)]^3 : \operatorname{div} v = 0 \text{ a.e. in } \Omega\}$. E is a (Hilbert) subspace of $[H_0^1(\Omega)]^3$ with the restricted norm and scalar product. We also denote $Du = [\partial_i u_j]_{1\leq i,j\leq 3}$ and assume that $\nu > 0$ is the viscosity constant.

Let b be the trilinear form defined on $[H_0^1(\Omega)]^3$ by

$$\begin{aligned} b(u,v,w) &= \int_\Omega \sum_{i,j=1}^{3} u_i(\partial_i v_j) w_j dx \\ &= \int_\Omega u^T (Du) w dx, \end{aligned}$$

for all $u, v, w \in [H_0^1(\Omega)]^3$.

We also assume that $j : V \to [0,\infty]$ is a convex, lower semicontinuous functional such that $j(0) = 0$, and $g : \Omega \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3,\ (x,u,\lambda) \mapsto g(x,u,\lambda)$ satisfies the Carathéodory condition (i.e. g_i satisfies this condition for each $i = 1,2,3$). We assume that g is differentiable with respect to u and $g, D_u g$ satisfies the usual growth condition:

$$\begin{cases} |g(x,u,\lambda)| \leq A(\lambda) + B(\lambda)|u|^{s-1} \\ |D_u g(x,u,\lambda)| \leq A(\lambda) + B(\lambda)|u|^{s-2}, \end{cases} \tag{2.24}$$

for a.e. $x \in \Omega$, all $u, \lambda \in \mathbb{R}$, with $A, B \in L^{\infty}_{loc}(\mathbb{R})$, $1 < s < 6(= 2^*)$.

Here u is the velocity of the fluid, b is the usual trilinear form in the Navier-Stokes equation, and g is the outer force acting on the fluid. g depends on u (in a nonlinear manner) and on λ, which usually represents the magnitude of the force. We assume that

$$g(x, 0, \lambda) = 0 \text{ for a.e. } x \in \Omega, \quad \text{all } \lambda \in \mathbb{R},$$

i.e., we have no external force at points with zero velocity. Here j is some kind of constraint imposed on the velocity. In many cases, j is of the form $j = I_K$, where K is a closed, convex subset of V, representing the set of admissible velocity fields of the fluid. For example, interesting choices of K are the following:

$$K = \{u \in E : u_1(x) \geq -c, \ u_2(x) \geq -d, \ c, d \geq 0\},$$

$$K = \{u \in E : |\nabla \times u| \leq c, \ c \geq 0\},$$

$$K = \{u \in E : |\int_S u \cdot n dS| \leq c, \ c \geq 0\}.$$

In the case $j = 0$, the variational inequality (2.23) becomes the equation:

$$\begin{cases} \nu \displaystyle\int_\Omega Du : Dv + b(u, u, v) = \int_\Omega g(x, u, \lambda) \cdot v, \ \forall v \in E \\ u \in E, \end{cases} \tag{2.25}$$

which is the usual variational form of the Navier-Stokes equation (cf. [11], [16], or [17]).

Other interesting choices for the functional j (the case of visco plastic Bingham fluids, cf. [11]) are:

$$j(u) = \int_\Omega \mu(x)|Du|^\gamma,$$

$$j(u) = \int_\Omega \mu(x)|\sum \epsilon_{ij}^2(u)|^\gamma,$$

where

$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$$

and μ is a nonnegative locally integrable function.

2.5 Bifurcation problems associated with the p-Laplace operator

In this example, we consider bifurcation problems for the following variational inequality:

$$\begin{cases} \displaystyle\int_\Omega |\nabla u|^{p-2}\nabla u \nabla(v-u) - \int_\Omega [\lambda |u|^{p-2}u + g(x, u, \lambda)](v-u) + j(v) - j(u) \\ \quad \geq 0, \ \forall v \in E, \\ u \in E. \end{cases} \tag{2.26}$$

Here $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary,

$$E = \{u \in W^{1,p}(\Omega) : v = 0 \text{ on } \Gamma\},$$

where Γ is a (relatively) open subset of $\partial\Omega$ with positive measure. $W^{1,p}(\Omega)$ is the usual Sobolev space, equipped with the norm,

$$\|u\|_{W^{1,p}(\Omega)} = \left[\int_\Omega (|u|^p + |\nabla u|^p)\right]^{1/p}, \ u \in W^{1,p}(\Omega).$$

$(E, \|\cdot\|_{W^{1,p}(\Omega)})$ is a closed (Banach) subspace of $W^{1,p}(\Omega)$. By Poincaré's inequality, we know that

$$\|u\| = \left(\int_\Omega |\nabla u|^p\right)^{1/p}, \ u \in E,$$

defines a norm on E, equivalent to $\|\cdot\|_{W^{1,p}(\Omega)}$. In the sequel, we will always consider E with this norm. We also define the pairing between E and E^* by $\langle \cdot, \cdot \rangle$. We assume that

$$g : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is a Carathéodory function, such that

$$g(x, u, \lambda) = o(|u|^{p-1}), \tag{2.27}$$

as $u \to 0$, uniformly a.e. with respect to $x \in \Omega$ and uniformly with respect to λ on bounded intervals, and, moreover, g satisfies the growth condition

$$|g(x, u, \lambda)| \leq C(\lambda)[m(x) + M|u|^{p-1}], \tag{2.28}$$

for a.e. $x \in \Omega$, all $u, \lambda \in \mathbb{R}$, where $C(\lambda) \geq 0$ is bounded on bounded sets, $m \in L^{\frac{p}{p-1}}(\Omega)$, and $M > 0$ is a constant.

As a particular choice for the functional j we shall take

$$j(u) = \int_{\partial\Omega} |u| dS, \ u \in V. \tag{2.29}$$

Other choices of j will also be considered.

3 The abstract setting

In this section we shall provide an abstract framework for a bifurcation analysis for the types of problems introduced in the previous section, section 2. The setting will be variational inequalities in reflexive Banach spaces.

3.1 Notation and definitions

Throughout we shall denote by E a reflexive Banach space and by E^* its dual. The norm in E will be denoted by $\|\cdot\|$ and that in E^* by $\|\cdot\|_*$. The pairing between E^* and E shall be given by $\langle\cdot,\cdot\rangle$, i.e. if $f \in E^*$ and $u \in E$, then $f(u) = \langle f, u\rangle$.

We shall assume that:

–

$$j, J : E \to \mathbb{R}_+ \cup \{\infty\}$$

are convex and lower semicontinuous functionals with

$$j(0) = J(0) = 0.$$

–

$$A, \alpha : E \to E^*$$

are continuous and bounded operators with

$$A(0) = \alpha(0) = 0,$$

which are strictly monotone, coercive and belong to class (S), i.e:

- *A is strictly monotone:*

$$\langle A(u) - A(v), u - v\rangle > 0, \text{ whenever } u \neq v.$$

- *A is coercive:* There exist constants $c > 0$ and $p > 1$ such that

$$\langle A(u), u\rangle \geq c\|u\|^p, \ \forall u \in E.$$

- *A belongs to class* (S) : For all weakly convergent sequences $\{v_n\}$, $v_n \rightharpoonup v$, with

$$\lim\langle A(v_n), v_n - v\rangle = 0,$$

it must hold that

$$v_n \to v.$$

–

$$B, f : \mathbb{R} \times E \to E^*$$

are completely continuous operators with

$$B(\lambda, 0) = 0 = f(\lambda, 0), \ \forall\lambda \in \mathbb{R}.$$

3.2 Homogenizations

The following relationships between the operators introduced above (section 3.1) will be assumed:

- For all sequences $\{v_n\}$, $v_n \to v$, and all sequences of positive numbers σ_n, $\sigma_n \to 0+$,
$$\lim \frac{1}{\sigma_n^{p-1}} A(\sigma_n v_n) = \alpha(v).$$
- For all weakly convergent sequences $\{v_n\}$, $v_n \rightharpoonup v$, and all sequences of positive numbers σ_n, $\sigma_n \to 0+$, all sequences $\{\lambda_n\}$, $\lambda_n \to \lambda$,
$$\lim \frac{1}{\sigma_n^{p-1}} B(\lambda_n, \sigma_n v_n) = f(\lambda, v).$$
- For all weakly convergent sequences $\{v_n\}$, $v_n \rightharpoonup v$, and all sequences of positive numbers σ_n, $\sigma_n \to 0+$,
$$\liminf \frac{1}{\sigma_n^{p}} j(\sigma_n v_n) \geq J(v),$$
further, for all $v \in E$, and all sequences of positive numbers σ_n, $\sigma_n \to 0+$, there exists a sequence $\{v_n\}$, $v_n \to v$, such that
$$\lim \frac{1}{\sigma_n^{p}} j(\sigma_n v_n) = J(v).$$

3.3 Equivalent operator equations

Consider, for $g \in E^*$, the variational inequality

$$\begin{cases} \langle A(u) - g, v - u\rangle + j(v) - j(u) \geq 0, \ \forall v \in E \\ u \in E. \end{cases} \tag{3.1}$$

It follows from classical results (see e.g. [7], [11]), that this problem is uniquely solvable, hence defines an operator

$$T_{A,j} : E^* \to E \tag{3.2}$$

by

$$T_{A,j}(g) = u,$$

where u is the unique solution of (3.1). This operator is also continuous (cf. [9]). Therefore, if we consider the variational inequality

$$\begin{cases} \langle A(u) - B(\lambda, u), v - u\rangle + j(v) - j(u) \geq 0, \ \forall v \in E, \\ u \in E, \end{cases} \tag{3.3}$$

then u solves (3.3) if and only if u solves

$$T_{A,j}B(\lambda, u) = u. \tag{3.4}$$

And similarly if we consider the variational inequality

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v - u \rangle + J(v) - J(u) \geq 0, \ \forall v \in E, \\ u \in E, \end{cases} \tag{3.5}$$

then u solves (3.5) if and only if u solves

$$T_{\alpha,J}f(\lambda, u) = u. \tag{3.6}$$

It follows from the relationships between A and α, B and f and j and J, that if u solves (3.5) then so does σu for any $\sigma(> 0) \in \mathbb{R}$.

3.4 Global bifurcation

Let us assume that $(\lambda_0, 0) \in \mathbb{R} \times E$ is a bifurcation point for (3.3), then it follows that (3.5) and hence also (3.6) will have a nontrivial solution for $\lambda = \lambda_0$. Therefore, if $a \in \mathbb{R}$ is such that (3.5) has only the trivial solution for $\lambda = a$, it will follow that for $r > 0$, sufficiently small, the Leray-Schauder degree

$$d(\mathrm{id} - T_{\alpha,J}f(a, \cdot), B_r(0), 0)$$

is defined (here $B_r(0)$ is the open ball of radius r in E centered at 0) and we obtain

$$d(\mathrm{id} - T_{\alpha,J}f(a, \cdot), B_r(0), 0) = d(\mathrm{id} - T_{A,j}B(a, \cdot), B_r(0), 0)$$

(see e.g. [9]). We hence may employ the homotopy invariance principle of the Leray-Schauder degree, to conclude that if $a, b \in \mathbb{R},\ a < b$ are such that (3.5) has only the trivial solution for $\lambda = a, b$ and if

$$d(\mathrm{id} - T_{\alpha,J}f(a, \cdot), B_r(0), 0) \neq d(\mathrm{id} - T_{\alpha,J}f(b, \cdot), B_r(0), 0) \tag{3.7}$$

then $[a, b] \times \{0\}$ will contain a bifurcation point for (3.4) and hence for (3.3) (cf. [8]). In fact, we may employ the global bifurcation result of Rabinowitz [15] to conclude that global bifurcation takes place in the sense of that theorem.

Thus in bifurcation problems of the type (3.3), in order to be able to apply the above considerations we need to compute the operators α and f, the functional J. Further one needs to find values $a, b \in \mathbb{R},\ a < b$ such that (3.7) holds for λ values a and b (by no means an easy task, in general). This we shall do for some of the examples considered in section 2 and refer the interested reader to many additional examples in [9].

4 Examples revisited

In this section we shall employ the abstract setting discussed in section 3 to discuss the existence of some bifurcation points for examples related to those introduced in section 2. We shall not dwell on the first example, since this problem is equivalent to the existence of bifurcation branches (in K) of four different nonlinear Sturm-Liouville problems, those problems being completely understood.

Before turning to the discussion of some of the other examples, we present some other abstract features common to some of them.

4.1 Semilinear problems

Let us assume

$$a : E \times E \to \mathbb{R}$$

is a continuous, coercive and bilinear form and let

$$A : E \to E^*$$

be defined by

$$\langle A(u), v \rangle = a(u, v).$$

Furthermore assume that

$$B(\lambda, u) = \lambda Bu + R(u), \ R(u) = o(\|u\|), \text{ as } u \to 0,$$

with B compact linear and that

$$j = I_K,$$

where K is a closed convex subset of E with $0 \in K$.

In this case one easily computes that $p = 2$, $\alpha = A$, $f(\lambda u) = \lambda Bu$ and $J = I_{K_0}$, where K_0 is the support cone of K, i.e

$$K_0 = \overline{\cup_{t>0} tK}.$$

If it is the case that K_0 is a subspace of E, then the variational inequality (3.5) becomes

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v - u \rangle + I_{K_0}(v) - I_{K_0}(u) \geq 0, \ \forall v \in E, \\ u \in E, \end{cases} \tag{4.1}$$

which is equivalent to

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v - u \rangle \geq 0, \ \forall v \in K_0 \\ u \in K_0, \end{cases} \tag{4.2}$$

and, since K_0 is a subspace, the latter is equivalent to

$$\begin{cases} \langle \alpha(u) - f(\lambda, u), v \rangle = 0, \ \forall v \in K_0 \\ u \in K_0. \end{cases} \tag{4.3}$$

From this we see (recall the comment at the end of section 3.3) that the solution operator $T_{\alpha,J}$ is a bounded linear operator and equation (3.6) becomes

$$u = \lambda T_{\alpha,J} B u. \tag{4.4}$$

Hence the possible bifurcation points for (3.3) are to be sought among the countable set $\{(\lambda_i, 0)\}$, where λ_i is a characteristic value of the compact linear operator $T_{\alpha,J}B$. And each characteristic value of odd multiplicity will yield a bifurcation point. We note here that what has just been said is true as long as J is the indicator function of a subspace, irregardless whether $j = I_K$ for some closed convex set K.

4.2 A semilinear elliptic problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, and let $\Gamma \subset \partial\Omega$ be a relatively open subset of positive measure.

Let

$$E = \{u \in H^1(\Omega) : u = 0, \text{ a.e. on } \Gamma\}.$$

Let

$$a : E \times E \to \mathbb{R}$$

be given by

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v,$$

then (because of Poincaré's inequality) a is a continuous, coercive and bilinear form. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with $g(u) = o(|u|)$ as $u \to 0$, and define $B(\lambda, u)$ by

$$\langle B(\lambda, u), v \rangle = \int_\Omega \lambda uv + g(u)v,$$

then

$$\langle f(\lambda, u), v \rangle = \int_\Omega \lambda uv.$$

Let us define the functional j by

$$j(u) = \int_{\partial\Omega} \mu |u|^\gamma,$$

where μ, γ are positive constants with $1 \le \gamma < 2$.

Embedding theorems (see [1], [6]) tell us that the mapping

$$H^1(\Omega) \hookrightarrow L^q(\partial\Omega)$$
$$u \mapsto u|_{\partial\Omega}$$

are compact for

$$1 \le q < \bar{p} = \begin{cases} \frac{2(N-1)}{N-2}, & N > 2 \\ \infty, & N = 1, 2. \end{cases}$$

It hence will follow that j is convex and lower semicontinuous and (since $p = 2$ and $1 \le \gamma < 2$) that

$$J(u) = I_{H_0^1(\Omega)}.$$

It hence follows from the results above, i.e. the results in section 4.1, that (3.5) is equivalent to the problem

$$\int_\Omega \nabla u \cdot \nabla v - \lambda \int_\Omega uv = 0, \ \forall v \in H_0^1(\Omega), \ u \in H_0^1(\Omega), \tag{4.5}$$

which is equivalent to the eigenvalue problem

$$\Delta u + \lambda u = 0, \ u \in H_0^1(\Omega). \tag{4.6}$$

We hence conclude that all eigenvalues of (4.6) which are of odd multiplicity yield bifurcation points.

4.3 An inequality involving the p-Laplacian

A situation, similar to the above, arises, if we consider the example presented in section 2.5. There we let

$$E = \{u \in W^{1,p}(\Omega) : u = 0, \text{ a.e. on } \Gamma\}$$

and let

$$A : E \to E^*$$

be given by

$$\langle A(u), v\rangle = \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v.$$

Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function with $g(u) = o(|u|^{p-1})$ as $u \to 0$, and define $B(\lambda, u)$ by

$$\langle B(\lambda, u), v\rangle = \int_\Omega \lambda |u|^{p-2}uv + g(u)v,$$

then

$$\langle f(\lambda, u), v\rangle = \int_\Omega \lambda |u|^{p-2}uv.$$

Let us define the functional j by

$$j(u) = \int_{\partial\Omega} \mu |u|,$$

where μ is a positive constant.

Again, using embedding theorems (see [1]) we see that the mapping

$$W^{1,p}(\Omega) \hookrightarrow L^1(\partial\Omega)$$
$$u \mapsto u|_{\partial\Omega}$$

is compact.

It hence will follow that j is convex and lower semicontinuous and that

$$J(u) = I_{W_0^{1,p}(\Omega)}.$$

It hence follows from the results above, i.e. the results in section 4.1, that (3.5) is equivalent to the problem

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla v - \lambda \int_\Omega |u|^{p-2} uv = 0, \ \forall v \in W_0^{1,p}\Omega, \tag{4.7}$$

which is equivalent to the eigenvalue problem

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) + \lambda |u|^{p-2} u = 0, \ u \in W_0^{1,p}(\Omega). \tag{4.8}$$

This eigenvalue problem has received much attention during recent years and several results about eigenvalues and the the computation of the Leray-Schauder degree of the associated completely continuous perturbation of the identity in a neighborhood of such eigenvalues have become available (see e.g. [2], [3], [4], [5]).

4.4 Stationary Navier-Stokes flows

In this section we consider the example discussed in section 2.4 and refer to this section for the statement of the problem and the notation.

Again the operator A is given by a continuous, coercive and bilinear form, hence $A = \alpha$. Also it easily follows that

$$\langle f(\lambda, u), v \rangle = \lambda \int_\Omega D_u g(x, 0) u \cdot v.$$

To hence obtain the homogeneous variational inequality (3.5) we must compute the functional J. To this end, we observe that if $j = I_K$, where K is any of the choices given in section 2.4, then $J = I_E$, since the support cone of K in any of the cases is the whole space.

We hence obtain that, in these cases, (3.5) is given by

$$\begin{cases} \nu \int_\Omega Du : Dv + \lambda \int_\Omega D_u g(x,0) u \cdot v = 0, \ \forall v \in E, \\ u \in E, \end{cases} \tag{4.9}$$

which is the eigenvalue problem for the Stokes equation. Its eigenvalues of odd multiplicity hence yield global bifurcation points for (2.23).

Let us now consider the case that j is given by

$$j(u) = \int_\Omega \mu(x) |Du|^\gamma,$$

where $\mu \in L^\infty(\Omega)$ and $\gamma \geq 1$. We observe that the effective domain of j is given by

$$D(j) = \{u : j(u) < \infty\} = \begin{cases} E, \ 1 \leq \gamma \leq 2 \\ \{u \in E : \mu |Du|^\gamma \in L^1(\Omega)\}, \ \gamma > 2 \end{cases}$$

Using these facts one may now compute

$$J = \begin{cases} I_W, \, 1 \leq \gamma < 2 \\ j, \, \gamma = 2 \\ I_E, \, \gamma > 2, \end{cases}$$

where

$$W = \{u \in E : Du = 0, \ \text{a.e. on } \Omega \setminus \Omega_0\},$$

and

$$\Omega_0 = \{x \in \Omega : \mu(x) = 0\}.$$

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