EQUADIFF 9 CD ROM, Brno 1997 PROCEEDINGS
Masaryk University pp. 25–48

Parabolic Equations with Multiple Singularities

Emmanuele DiBenedetto*

Department of Mathematics
Northwestern Univ. Evanston ILL. 60208–2730, USA
Email: dibenede@giotto.math.nwu.edu

Abstract. Models of flow of multiple immiscible fluids in a porous matrix and/or phenomena of multiple transitions of phase, result into quasilinear parabolic equations, with measurable coefficients and exhibiting multiple singularities and/or degeneracies (in the sense made precise in Section 1.1 below). We discuss the problem of the continuity of the transition parameters, for example saturation in the flow of immiscible fluids, or temperature in isothermal phase transitions. We review and summarize the main points of the theory and will present some recent results in this direction, pointing to the new mathematical tools generated by these investigations. We will also indicate the main open questions of physical and mathematical interest and discuss their relevance.

AMS Subject Classification. 35K20, 35K40, 35K55, 35K65, 35K99, 35Q35

Keywords. Degenerate Parabolic Equations, Singular parabolic equations, immiscible fluids, phase transition, Harnack estimates, Hadamard growth, modulus of continuity, DeGiorgi estimates, intrinsic geometry, measure theory

1 Introduction

We will present some recent results concerning the local behavior of weak solutions of singular parabolic equations with measurable coefficients. We will indicate the main points of the theory and will trace back their motivation to physical phenomena, such as transition of phase and/or flow of immiscible fluids in a porous matrix. In this connection, we will also indicate some novel analytical ideas of measure theory which we feel are of independent interest. Along the presentation we will point out the main open problems, which we feel have both a theoretical and physical interest.

* Partially supported by NSF Grant DMS–9706388

1.1 Singular Parabolic Equations

Let $\beta(\cdot)$ be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$ and consider parabolic inclusions of the type

$$\frac{\partial}{\partial t}\beta(u) - \operatorname{div}\mathbf{A}(x,t,u,\nabla u) + B(x,t,u,\nabla u) \ni 0 \quad \text{in } \Omega_T, \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N$ and ∇ denotes the gradient with respect to the space variables only. Also, for $T>0$ we have set $\Omega_T \equiv \Omega\times(0,T]$. We assume that the graph $\beta(\cdot)$ is coercive, i.e., there exists a positive constant γ_o, such that for all pairs of real numbers $(s_1,\, s_2)$ and all selections $w_1\in\beta(s_1)$ and $w_2\in\beta(s_2)$,

$$w_1 - w_2 \geq \gamma_o(s_1 - s_2). \tag{1.2}$$

We also assume that $\beta(\cdot)$ is bounded for bounded values of its argument, i.e.,

$$\text{for every } M>0, \qquad \sup_{|s|<M}\ \sup_{w\in\beta(s)} |w| < \infty. \tag{1.3}$$

No further condition is formulated on the behavior of $\beta(\cdot)$. In particular in any finite interval $(-M,M)$, the graph $\beta(\cdot)$ might exhibit countably many jumps or might become vertical countably many times, exponentially fast or faster. If a graph $\beta(\cdot)$ exhibits this behavior we call it a *singular* graph and refer to (1.1) as singular parabolic equations. Examples of such a $\beta(\cdot)$ are

$$\beta(s) \equiv \begin{cases} s & \text{if } s<0, \\ [0,1] & \text{if } s=0, \\ 1+s & \text{if } s>0; \end{cases} \qquad \beta(s) \equiv \begin{cases} 2+s & \text{if } s>1, \\ [2,3] & \text{if } s=1, \\ 1+s & \text{if } 0<s<1, \\ [0,1] & \text{if } s=0, \\ s & \text{if } s<0; \end{cases} \tag{i}$$

$$\begin{aligned} &\beta(s) \equiv |s|^{\frac{1}{m}}\operatorname{sign} s, \qquad m>1; \\ &\beta(s) \equiv 1 + s^{\alpha_1} - (1-s)^{\alpha_2}, \quad \begin{cases} s\in[0,1], \\ \alpha_i \in (0,1), i=1,2. \end{cases} \end{aligned} \tag{ii}$$

The first of (**i**) is the enthalphy function in the weak formulation of a Stefan-like problem modeling a water-ice transition of phase.[1] The second might serve as a prototype of the enthalpy in a double transition of phase. The first of (**ii**) is the graph arising from the classical porous media equation, modeling the flows

[1] There exists a vast literature on each of the several aspects of the classical Stefan problem. For a summary of the main results we refer to the monograph of Meirmanov [42], the review article of Danilyuk [13] as well as the Proceedings [8,26,30] and the references therein. Here we review only those aspects connected with the local continuity of weak solutions of (1.1) with $\beta(\cdot)$ exhibiting multiple singularities.

of a *single* fluid in a porous matrix.[2] The second, is a first approximation for a model of two immiscible fluids moving within a porous matrix.[3] The simplest example of (1.1) is,

$$\frac{\partial}{\partial t}\beta(u) - \Delta u \ni 0 \quad \text{in} \quad \Omega_T. \tag{1.1$'$}$$

The diffusion field $\mathbf{A}$ and the forcing term B in (1.1), are real valued and measurable over $\Omega_T \times \mathbb{R} \times \mathbb{R}^N$, and satisfy the structure conditions,

$$\begin{aligned} \mathbf{A}(x,t,\eta,\xi)\cdot\xi &\geq \mu_o\,|\xi|^2 - \varphi_o(x,t); \\ |\mathbf{A}(x,t,\eta,\xi)| &\leq \mu_1\,|\xi| - \varphi_1(x,t); \\ |B(x,t,\eta,\xi)| &\leq \mu_2\,|\xi|^2 - \varphi_2(x,t), \end{aligned} \tag{1.4}$$

for a.e. $(x,t,\eta,\xi) \in \Omega_T \times \mathbb{R} \times \mathbb{R}^N$. Here μ_i, $i = 0,1,2$ are prescribed positive numbers and φ_i, $i = 0,2$ are prescribed nonnegative functions defined a.e. in Ω_T, satisfying

$$\varphi_o + \varphi_1^2 + \varphi_2 \in L^{\overline{q},\overline{r}}_{\text{loc}}(\Omega_T). \tag{1.5}$$

The numbers $\overline{q}$ and $\overline{r}$ are positive, are linked by

$$\frac{1}{\overline{r}} + \frac{N}{\overline{q}} = 1 - \overline{\kappa}, \qquad \overline{\kappa} \in (0,1), \tag{1.6}$$

and can be taken out of their admissible range

$$\begin{aligned} &\overline{q} \in \left[\tfrac{N}{2(1-\overline{\kappa})}\,,\,\infty\right], && \overline{r} \in \left[\tfrac{1}{1-\overline{\kappa}}\,,\,\infty\right], && 0<\overline{\kappa}<1, && \text{for } N \geq 2; \\ &\overline{q} \in (1\,,\,\infty), && \overline{r} \in \left[\tfrac{1}{1-\overline{\kappa}}\,,\,\tfrac{1}{1-2\overline{\kappa}}\right], && 0<\overline{\kappa}<\tfrac{1}{2}, && \text{for } N = 1. \end{aligned} \tag{1.7}$$

The inclusion in (1.1) is meant weakly and in the sense of graphs. Precisely, a function

$$u \in L^2_{\text{loc}}\left\{0,T;W^{1,2}_{\text{loc}}(\Omega)\right\}, \tag{1.8}$$

is a local weak solution to (1.1) if there exists a measurable selection $w \subset \beta(u)$, such that

$$t \to w(\cdot,t) \text{ is weakly continuous in } L^2_{\text{loc}}(\Omega), \tag{1.9}$$

[2] Also the porous medium equation has been widely investigated in the literature and we refer to the same Proceedings [8,26,30] and their references, for an overview. An overview of the main results regarding the local regularity of the solutions, is in the Bibliographical Notes of the monograph [22].

[3] A 1-dimensional model in hydrology is investigated by Van Duijn and Zhang in [15], and numerically by Hoff [29]. Most of the models of multiphase flows in porous medium are multidimensional. For such models we refer to the monographs [6,7,11,12,49].

and in addition,

$$\int_\Omega w(x,t)\varphi(x,\tau)\,dx\Big|_{\tau=t_1}^{\tau=t_2} + \int_{t_1}^{t_2}\int_\Omega \big\{ - w(x,\tau)\varphi_t + \mathbf{A}(x,\tau,u,\nabla u)\cdot D\varphi\big\}\,dxd\tau + \int_{t_1}^{t_2}\int_\Omega B(x,\tau,u,\nabla u)\varphi\,dxd\tau = 0, \tag{1.10}$$

for all testing functions

$$\varphi \in W^{1,2}_{\mathrm{loc}}\left\{0,T;L^2_{\mathrm{loc}}(\Omega)\right\} \cap L^2_{\mathrm{loc}}\left\{0,T;W^{1,2}_o(\Omega)\right\}, \tag{1.11}$$

and for all intervals $(t_1,t_2)\subset(0,T]$.

2 The Problem of Continuity of Weak Solutions

It is natural to ask whether *locally bounded* weak solutions to (1.1) are continuous in Ω_T and whether one can estimate *quantitatively* their modulus of continuity. To simplify the setting of the problem, we assume that u is a solution of (1.1) bounded in the whole Ω_T and set,

$$\|u\|_{\infty,\Omega_T} \equiv M. \tag{2.1}$$

This is not restrictive, by regarding Ω_T as a subset of the domain of definition of u. By the same token we also assume that the integrability requirement in (1.5) holds in Ω_T and set,

$$\left\|\varphi_o+\varphi_1^2+\varphi_2\right\|_{\bar{q},\bar{r};\Omega_T} \equiv \Phi. \tag{2.2}$$

We refer to the numbers,

$$N, \quad \gamma_o, \quad M, \quad \Phi, \quad \mu_i, i=0,1,2,$$

as the *data*. For a constant C or γ, or a *continuous* function $\omega(\cdot)$ we say

$$C\equiv C(\text{data}), \quad \gamma\equiv\gamma(\text{data}), \quad \omega(\cdot)=\omega_{\text{data}}(\cdot),$$

if they can be determined a priori only in terms of the indicated parameters. Having fixed an arbitrary subset $\mathcal{K}\subset\Omega_T$, one can ask whether u is continuous in $\mathcal{K}$ with a modulus of continuity $\omega_{\text{data}}(\cdot)$ depending only upon the data and the distance from $\mathcal{K}$ to the parabolic boundary of Ω_T.

Remark 2.1. If $\beta(\cdot)\equiv\mathbb{I}$, then locally bounded solutions of (1.1) are locally Hölder continuous in Ω_T, and the assumptions (1.5)–(1.7) are optimal for this to occur.[4] Thus the issue at hand is to investigate to what extend the singularity of $\beta(\cdot)$ might affect the continuity of u.

[4] For a general account of the theory of local regularity of solutions of non-singular parabolic equations with measurable coefficients, we refer to the monograph [39], and in particular Chap. I, §3,4; Chap. II, §6,7; Chap. V, §1,2.

Remark 2.2. The assumption that u be locally bounded is essential. Indeed even if $\beta(\cdot) \equiv \mathbb{I}$, weak solutions of (1.1) need not be bounded. This is due to the critical growth of the forcing term $B(x,t,u,\nabla u)$ with respect to $|\nabla u|$ as indicated in the last of (1.4). We refer Stampacchia [51] for counterexamples even in the elliptic case.[5] If the last of (1.4) were replaced with

$$|B(x,t,\eta,\xi)| \leq \mu_2\,|\xi| + \varphi_2(x,t), \tag{1.4$'$}$$

then weak solutions of (1.1), for any coercive $\beta(\cdot)$ as in (1.2)–(1.3) would be locally bounded. This would follow from a simple adaptation of the methods of [39].[6]

In what follows we will assume in addition that the local solution u can be constructed as the limit in the topology of (1.8), of a sequence of smooth local solutions of (1.1) for *smooth* $\beta(\cdot)$. This assumption is formulated only to justify some of the calculations.[7] We stress that the modulus of continuity of u must be independent of any approximating procedure and must depend only upon the data.

3 Some Degenerate Parabolic Equations

The full generality indicated in (1.4)–(1.6) seems to be natural in physical models, such as the simultaneous flow of two immiscible fluids in a porous matrix.[8] These models typically lead to degenerate parabolic equations of the type,[9]

$$v_t - \operatorname{div}\mathbf{a}\,(x,t,v,\nabla v) + b\,(x,t,v,\nabla v) = 0 \quad \text{in } \Omega_T. \tag{3.1}$$

[5] Thus (1.1) even with $\beta(\cdot) = \mathbb{I}$ might have unbounded solutions. However if one had some a priori qualitative knowledge of the boundedness of the solution, such a qualitative bound could be turned into a quantitative one. See for example Vespri [53] and references therein.

[6] Indeed a slightly faster growth is allowed; for example $|\nabla u|^q$ where $0 \leq q < \frac{N+4}{N+2}$. See [39] Chap. V, §1.

[7] If the forcing term $B(x,t,u,\nabla u)$ has at most a linear growth with respect to $|\nabla u|$, then questions of existence and uniqueness are well understood. We refer for example to the monographs [27,39,41] and the Proceedings [8,26,30] and references therein. Here we only remark that a modulus of continuity uniform with respect to the approximating procedure, would supply the necessary compactness to establish existence of solutions.

[8] For these models we refer to the monographs of J. Bear [6] (Chap. 9) and [7] (Chap. 6), R. E. Collins [12] (Chap. 6), and A.E. Scheidegger [49] (Chap. 10), and the article of Leverett [40]. These models consist of a system of two parabolic equations, written in terms of the saturations and pressures of each of the two fluids.

[9] The transformation of Kruzkov-Sukorjanski [37], transforms the physical models of [6,7,12,40,49] into a system of one parabolic equation like (3.1) in terms of the saturation v of only one of the two fluids, and another degenerate-elliptic equation in terms of a mean pressure. In such a formulation, the term $b(x,t,v,\nabla v)$ in (3.1) would depend on such a mean pressure. The local continuity for the saturations was first raised in [1] and [21]. The analysis of [1,21] permits to reduce the question of the continuity of the saturations to the continuity of solutions to (3.1).

The leading vector field $\mathbf{a}$ and the forcing term b, are measurable and satisfy,

$$\begin{aligned} \mathbf{a}(x,t,v,\nabla v)\cdot\nabla v &\geq C_o\varphi(v)|\nabla v|^2-\varphi_o(x,t);\\ |\mathbf{a}(x,t,v,\nabla v)| &\leq C_1\varphi(v)|\nabla v|-\varphi_1(x,t);\\ |b(x,t,v,\nabla v)| &\leq C_2\varphi(v)|\nabla v|^2-\varphi_2(x,t), \end{aligned} \tag{3.2}$$

for a.e. $(x,t)\in\Omega_T$ and all smooth functions $(x,t)\to v(x,t)$ defined in Ω_T. Because of the physical origin of the p.d.e., it is natural to assume that the solutions are bounded, say for example $v\in[0,1]$.[10] The equation is degenerate in the sense that $\varphi(\cdot)$ is permitted to vanish. Precisely we assume that $v\to\varphi(v)$ is continuous, non-negative and vanishes at the extreme values of its argument, i.e.,

$$\varphi(v)>0 \text{ for } v\in(0,1) \text{ and } \varphi(0)=\varphi(1)=0. \tag{3.3}$$

The functions φ_i, $i=0,1,2$ satisfy the assumptions (1.5)–(1.7). A notion of solution to (3.1) is introduced along the lines of (1.8)–(1.11), by requiring that $t\to v(\cdot,t)$ satisfies (1.9) and that

$$\nabla\varphi(v)\in L^2_{\mathrm{loc}}(\Omega_T).$$

The main difficulty in establishing the local continuity of v resides in the double degeneracy of $\varphi(\cdot)$ and, more importantly, in the lack of precise quantitative and/or qualitative information on its modulus of continuity. Such a limited information on the nature of the degeneracy is typical of the physical models of flows of a mixture of fluids in a porous medium.[11] Thus in particular $\varphi(\cdot)$ might degenerate at $v=0$ and $v=1$ at different rates, and perhaps exponentially fast or faster.[12] The problem of continuity of weak solutions to (3.1) consists in showing that v is continuous *whatever* the nature of the degeneracy of $\varphi(\cdot)$, provided (3.3) is satisfied.

Let $u\in[0,1]$ be a solution of (1.1) with $\beta(\cdot)\in C(0,1)$ and singular at the extreme values $u=0$ and $u=1$ of its argument, i.e. for example

$$\lim_{u\searrow 0}\beta'(u)=\lim_{u\nearrow 1}\beta'(u)=+\infty.$$

[10] The function $(x,t)\to v(x,t)$ is the local relative saturation of one of the two fluids. Thus $v\in[0,1]$. See for example [1,6,7,12,37,49].

[11] The function $\varphi(\cdot)$ is related to the permeability of both fluids. The permeability of one of the fluids vanishes as the fluid is displaced by the other (i.e., either $v=0$ or $v=1$). This is the physical origin of the degeneracy of $\varphi(\cdot)$. The behaviour of the permeabilities as functions of the saturations are derived from hydrostatic (rather than dynamic) experiments, [6,7,12,49], dimensional analysis [40], and heuristic arguments. For this reason the information on their rate of vanishing is rather limited.

[12] In fact, because of the phenomenon of the connate water it might be even completely flat in a small right neighborhood of zero or a left interval of 1, or both. See Bear [6] Chap. 9, §2.3 and 2.4; Collins [12] Chap. 2, §24, and Chap.6, §10; Scheidegger [49] Chap. 3, §4 and Chap.10, §6.

Then by setting $v \equiv \beta(u)$ and $\varphi(\cdot) \equiv \beta^{-1\prime}(\cdot)$ the singular p.d.e. in (1.1), in terms of u, can be recast as the degenerate p.d.e. in (3.1), in terms of v, and one checks that the conditions (1.4) yield (3.2). For this reason, the methods introduced in the context of (1.1) and those connected to (3.1) bear a considerable similarity and/or overlap. Starting now from (3.1) one might set

$$u \equiv \int^{v} \varphi(s)\,ds, \qquad \beta(u) = v,$$

and attempt to recast the degenerate p.d.e. (3.1) as the singular equation (1.1). One verifies that the resulting leading coefficients $\mathbf{A}$ would satisfy the first two of (1.4). The resulting free term B however might not satisfy the last (1.4), due to its faster than linear growth with respect to $|\nabla v|$.[13] In what follows we will outline the analogies and point to the main differences.

4 The Classical Approach to Continuity

For positive ρ, let K_ρ and Q_ρ denote respectively the cube of wedge 2ρ centered at the origin of $\mathbb{R}^N$, and the parabolic cylinder with "vertex" at the origin of $\mathbb{R}^{N+1}$, with cross sections K_ρ, i.e.,

$$K_\rho \equiv \{x \in \mathbb{R}^N | \max_{1\le i\le N} |x_i| < \rho\}, \qquad Q_\rho \equiv K_\rho \times (-\rho^2, 0). \tag{4.1}$$

A cube centered at some $x_o \in \mathbb{R}^N \backslash \{0\}$ and congruent to K_ρ will be denoted by $\{x_o + K_\rho\}$ and a parabolic cylinder with "vertex" at some $(x_o, t_o) \in \mathbb{R}^{N+1}$ and congruent to Q_ρ, will be denoted by $\{(x_o, t_o) + Q_\rho\}$. In what follows we will fix a point $(x_o, t_o) \in \Omega_T$ and let ρ_o be the largest radius so that $\{(x_o, t_o) + Q_{\rho_o}\}$ is contained in Ω_T. Also for a constant $\delta \in (0, 1)$ we consider the sequence of decreasing radii,

$$\rho_n \equiv \delta^n \rho_o, \qquad n = 0, 1, 2 \ldots, \tag{4.2}$$

and the family of nested shrinking cylinders, with the same vertex at (x_o, t_o),

$$\{(x_o, t_o) + Q_{\rho_n}\}, \qquad n = 0, 1, 2, \ldots .$$

4.1 Non Singular Parabolic Equations

Suppose for the moment that in (1.1), the graph $\beta(\cdot)$ is the *identity*, i.e., that (1.1) is a quasilinear, *non-singular*, parabolic equation with measurable coefficients. If u is a weak solution to such an equation, we set

$$\mu_n^+ \equiv \operatorname*{ess\,sup}_{\{(x_o,t_o)+Q_{\rho_n}\}} u, \quad \mu_n^- \equiv \operatorname*{ess\,inf}_{\{(x_o,t_o)+Q_{\rho_n}\}} u, \quad \omega_n \equiv \operatorname*{ess\,osc}_{\{(x_o,t_o)+Q_{\rho}\}} u.$$

[13] One verifies this for the equation $v_t - \Delta v^2 = v|\nabla v|^2$. The equivalence of the two formulations would hold if B had a linear growth with respect to $|\nabla u|$. Equations such as (3.1) arising from the flow of immiscible fluids in a porous medium bear lower order terms with a behavior technically similar to a super-linear growth with respect to $|\nabla v|$. See [1], §3–6.

Proposition 4.1. *Let u be a weak solution of (1.1) with $\beta(\cdot) \equiv \mathbb{I}$. Then there exists constants $C > 1$ and $\delta, \eta \in \left(0, \frac{1}{2}\right)$ that can be determined a priori only in terms of the data, such that for every $(x_o, t_o) \in \Omega_T$,*

$$\omega_{n+1} \leq (1-\eta)\omega_n + C\rho_n^\lambda, \quad n = 0, 1, 2, \dots . \tag{4.3}$$

Here $\lambda \in (0,1)$ is a number determined only in terms of the integrability conditions (1.5)–(1.7) and is independent of δ and η. As a consequence u is locally Hölder continuous in Ω_T.

Proof of Hölder continuity assuming (4.3). Having fixed $(x_o, t_o) \in \Omega_T$, from (4.3) by iteration we derive,

$$\omega_n \leq (1-\eta)^n \omega_o + \frac{C}{\delta^\lambda} \sum_{i=1}^{n} \left(\frac{1-\eta}{\delta^\lambda} \right)^n, \quad \forall n \in \mathbb{N}. \tag{4.4}$$

The two numbers $(1-\eta)$ and δ can be related by

$$(1-\eta) = \delta^\alpha, \qquad \text{where} \qquad \alpha = \frac{\ln(1-\eta)}{\ln \delta} \in (0,1).$$

Moreover without loss of generality we may assume that $\rho_o \in (0,1)$. Then, having determined δ and η, the iterative inequalities (4.3) continue to hold if λ is replaced by a smaller number. We will choose it so that $(1-\eta)\delta^{-\lambda} < 1$. This way the sum on the right hand side of (4.4) can be majorized with a convergent series. Therefore from (4.4) and the definition (4.2) of the sequence ρ_n, it follows that,

$$\omega_n \leq \omega_o \left(\frac{\rho_n}{\rho_o} \right)^\alpha + \gamma(\text{data}; \eta, \delta)\rho_n^\lambda, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Since $(x_o, t_o) \in \Omega_T$ is arbitrary, this implies that u is locally Hölder continuous in Ω_T, with Hölder exponent $\min\{\alpha; \lambda\}$. □

Remark 4.1. Having fixed $(x_o, t_o) \in \Omega_T$, the starting cylinder $\{(x_o, t_o) + Q_{\rho_o}\}$ must be contained in Ω_T. Thus from the form of (4.5) it follows that the Hölder continuity can be claimed only within compact subsets $\mathcal{K}$ of Ω_T and that the Hölder constant $\omega_o \rho_o^{-\alpha}$ deteriorates as (x_o, t_o) approaches the parabolic boundary of Ω_T.

Remark 4.2. The constant C appearing on the right hand side of (4.3) is due only to the functions φ_i in the structure conditions (1.4) and it would be zero for the prototype equation (1.1′), with $\beta(\cdot) \equiv \mathbb{I}$.

This is the parabolic version of the classical DeGiorgi's approach to continuity introduced in the context of elliptic equations with measurable coefficients [14]. The adaptation to parabolic equations is far from simple and it appears in the book [39]. The same point of view of reducing the oscillation of u in a family of *shrinking cylinders* has influenced, one way or another, most of the literature on the subject, including Moser [43,44], Trudinger [52], Kruzkov [34,35,36], Aronson-Serrin [5] and Krylov-Safonov [38]. The reduction of the oscillation (4.3) is realized by the following Proposition that can be regarded as some sort of a weak maximum principle.[14]

Proposition 4.2. *Let u be a weak solution of (1.1) with $\beta(\cdot) \equiv \mathbb{I}$. Then there exists constants $C > 1$ and $\delta, \eta \in \big(0, \frac{1}{2}\big)$, that can be determined a priori only in terms of the data, such that for every $(x_o, t_o) \in \Omega_T$ and every $n \in \mathbb{N}$, either $\omega_n < C\rho_n^\lambda$, or at least one of the following two inequalities holds,*

$$\begin{aligned} u(x,t) &\le \mu_n^+ - \eta\omega_n, \\ u(x,t) &\ge \mu_n^- + \eta\omega_n, \end{aligned} \qquad \textit{for a.e. } (x,t) \in \big\{(x_o,t_o) + Q_{\rho_{n+1}}\big\}. \tag{4.6}$$

Proof of (4.3) assuming (4.6). Fix $(x_o, t_o) \in \Omega_T$ and $n \in \mathbb{N}$. If the first of (4.6) holds true, then

$$\operatorname*{ess\,sup}_{\{(x_o,t_o)+Q_{\rho_{n+1}}\}} u = \mu_{n+1}^+ \le \mu_n^+ - \eta\omega_n.$$

Subtracting μ_{n+1}^- from the left hand side and μ_n^- from the right hand side, gives

$$\omega_{n+1} = \mu_{n+1}^+ - \mu_{n+1}^- \le \mu_n^+ - \mu_n^- - \eta\omega_n = (1-\eta)\omega_n.$$

A similar argument proves the claim if the second of (4.6) holds. □

The proof of this Proposition is in [39] and is a parabolic version of a similar *elliptic* Proposition proved by DeGiorgi [14].

5 Parabolic Equations with One-Point Singularity

We consider now (1.1) where $\beta(\cdot)$ is singular at *only one point*. This would include the Stefan graph indicated in the first of (**i**) and the porous medium graph indicated in the first of (**ii**).

The first regularity results for weak solutions of these equations with such a $\beta(\cdot)$, appear in [9,18,19,47,48,55]. In all these contributions, the basic approach to continuity is analogous to that of Propositions 4.1–4.2. The proofs differ essentially from the technical ways of establishing an alternative similar to that in Proposition 4.2. The singularity of $\beta(\cdot)$ affects Proposition 4.2 in two ways, i.e., the reduction factor δ that determines the sequence of radii ρ_n in (4.2), and the number η that determines the reduction of the oscillation in (4.3), are both functions of the oscillation itself. Given two continuous, monotone increasing functions

$$(0, 2M] \ni s \to \delta(s), \eta(s) \in (0,1), \quad \text{such that} \quad \delta(0) = \eta(0) = 0, \tag{5.1}$$

[14] Suppose the first of (4.4) holds and assume without loss of generality that (x_o, t_o) coincides with the origin of $\mathbb{R}^{N+1}$. Then the $\sup u$ over the smaller cylinder $Q_{\rho_{n+1}}$ is strictly less than the $\sup u$ over the larger and coaxial cylinder Q_{ρ_n}. Thus the $\sup u$ over the larger cylinder can only be achieved in the parabolic shell $Q_{\rho_n} \backslash Q_{\rho_{n+1}}$. This can be regarded as some sort of parabolic boundary for the larger box.

we construct inductively, the decreasing sequences of numbers

$$\begin{aligned} \omega_o &= \max\left\{2M\,;\,C\rho_o^{\lambda}\right\}, \\ \rho_{n+1} &= \delta(\omega_n)\,\rho_n, \qquad\qquad\qquad \forall n\in\mathbb{N}, \\ \omega_{n+1} &= \max\left\{\big(1-\eta(\omega_n)\big)\omega_n ; C\rho_n^{\lambda}\right\} \end{aligned} \tag{5.2}$$

and the corresponding family of shrinking nested cylinders $\{(x_o,t_o)+Q_{\rho_n}\}$, with the same "vertex" at (x_o,t_o). Here $C>1$ and $\lambda\in(0,1)$ are two given constants.

Lemma 5.1. $\{\omega_n\}\to 0$ *as* $n\to\infty$.

Proof. From the definition it follows that the sequences $\{\rho_n\}$ and $\{\omega_n\}$ are non-increasing, so that their limits as $n\nearrow\infty$ exist. Since $\delta(\cdot)\in(0,1)$, it is apparent that $\{\rho_n\}\searrow 0$. If

$$\lim_{n\to\infty}\omega_n=\omega_\infty>0,$$

then, using the monotonicity of $\eta(\cdot)$, we derive from (5.2),

$$\omega_{n+1}\le\max\left\{\big(1-\eta(\omega_\infty)\big)\omega_n ; C\delta^{\lambda n}(\omega_\infty)\rho^{\lambda}\right\},\quad\forall n\in\mathbb{N}.$$

Thus $\{\omega_n\}\searrow 0$ against the contradiction assumption. □

Proposition 5.1. *Let u be a weak solution of (1.1) with $\beta(\cdot)$ either of Stefan-type (i.e., the first of (***i***)) or of the type of porous media (i.e., the first of (***ii***)). Then there exist constants $C>1$ and $\lambda\in(0,1)$, and two continuous increasing functions $\delta(\cdot)$ and $\eta(\cdot)$ as in (5.2), that can be determined a priori only in terms of the data, such that for every $(x_o,t_o)\in\Omega_T$,*

$$\operatorname*{ess\,osc}_{\{(x_o,t_o)+Q_{\rho_n}\}} u\le\omega_n \qquad n=0,1,2,\dots. \tag{5.3}$$

Here $\lambda\in(0,1)$ is a number determined only in terms of the integrability conditions (1.5)–(1.7) and is independent of δ and η. As a consequence u is locally continuous in Ω_T.

Remark 5.1. The constants C and λ in (5.2), depend only upon the functions φ_i, $i=0,1,2$ in the structure conditions (1.4) and can be taken to be zero for the prototype equation (1.1′).

Remark 5.2. While Proposition 4.1 implies a precise Hölder modulus of continuity, this is not longer the case for Proposition 5.1. The sequences (5.2) and the recursive bound (5.3) supply a *quantitative* but *not explicit* modulus of continuity for u.[15]

[15] In the case of graphs of Stefan-type the functions $s\to\delta(s)$, $\eta(s)$ have the explicit form $K^{-h/s}$ where K and h are large constants (see [18]). It would be of interest to generate an explicit modulus of continuity for u, in terms of K and h.

The contributions in [9,18,19,47,48,55] all establish recursive inequalities similar to those of Proposition 5.1, even though with technically different points of view. In [18,19] the Proposition was established by means of DeGiorgi-type iterations, in the parabolic setting of [39]. The proof of [55] follows the Harnack-type techniques of Moser, as appearing in [43,44,5,52]. The results of [9] make use of local representations in terms of heat potentials, and for this reason are limited to the prototype equations (1.1′). The results of [47,48] are also limited to (1.1′), being based on the non-divergence structure *shrinking technique* of Krylov and Safonov [38].

Whatever the approach however, it is essential that $\beta(\cdot)$ be singular at *only one point*, say for example at $u = 0$.

All these proofs have a common pattern, i.e., having fixed a cylinder $\{(x_o, t_o) + Q_\rho\}$, either the singularity occupies a small portion of such a box or a large one. The first case is a *favorable*, in the sense that the singularity plays a negligible role. If the second case occurs, then since off the singular set the partial differential equation (1.1) is uniformly parabolic, the solution cannot grow too fast and remains "close" to a fixed value, for example μ^+, and it does not oscillate too much. This supplies a control on the oscillation which in turn can be rephrased as in Proposition 5.1.

Technically, the solution u remains "close" to μ^+ within $\{(x_o, t_o) + Q_\rho\}$, if the functions

$$(u - k)_+ \equiv \max\{u - k; 0\}, \qquad 0 < k < \mu^+,$$

are subsolutions of a *uniformly parabolic* equation. This in turn is possible if, for $u \geq k > 0$, the graph $\beta(\cdot)$ does not suffer any other singularity. By working with the infimum μ^- a similar argument indicates that $\beta(\cdot)$ cannot have a singularity for $u < 0$. Thus the only singularity permitted is at a single point. This is the main limitation of these proofs.

6 Power-Like One Point Singularity

Consider now (1.1) with $\beta(\cdot)$ given by the first of (**ii**). In such a case we rewrite the p.d.e. as

$$|u|^{\frac{1-m}{m}} u_t - \operatorname{div} \mathbf{A}(x, t, u, \nabla u) + B(x, t, u, \nabla u) = 0 \quad \text{ in } \Omega_T. \tag{6.1}$$

If the coefficient of u_t were constant, one might perform a change of the time variable to transform (6.1) into a *non-singular*, uniformly parabolic equation. Following this remark, one might introduce an *intrinsic time scale* in the parabolic cylinders $\{(x_o, t_o) + Q_\rho\}$, with respect to which (6.1) would exhibits properties typical of uniformly parabolic equations. This idea has been introduced and implemented in [23]. The new *intrinsic geometry* is constructed as follows. For $\omega > 0$ let $Q_\rho(\omega)$ denote the cylindrical domain with "vertex" at the origin of $\mathbb{R}^{N+1}$,

$$Q_\rho(\omega) \equiv K_\rho \times \{-\rho^2 \omega^{\frac{1-m}{m}}, 0\}. \tag{6.2}$$

For $(x_o, t_o) \in \Omega_T$, we let $\{(x_o, t_o) + Q_\rho(\omega)\}$ denote the cylinder congruent to $Q_\rho(\omega)$ and with "vertex" at (x_o, t_o). Having fixed $(x_o, t_o) \in \Omega_T$ and $\omega > 0$, we will choose $\rho > 0$ so that $\{(x_o, t_o) + Q_\rho(\omega)\} \subset \Omega_T$. Let us fix two *constants* δ, $\eta \in \left(0, \frac{1}{2}\right)$ satisfying

$$\delta < (1-\eta)^{\frac{m-1}{m}}, \tag{6.3}$$

and construct, inductively, sequences $\{\omega_n\}$, $\{\rho_n\}$ and a family of nested and shrinking cylinders as follows.

$$\begin{aligned} &\omega_o = 2M, \rho_o \text{ such that } \{(x_o, t_o) + Q_{\rho_o}(\omega_o)\} \subset \Omega_T; \\ &\omega_{n+1} = (1-\eta)\omega_n + C\rho_n^\lambda, \quad \rho_{n+1} = \delta^n \rho_o, \quad \forall n \in \mathbb{N}; \\ &\{(x_o, t_o) + Q_{\rho_n}(\omega_n)\}. \end{aligned} \tag{6.4}$$

Here $C > 1$ and $\lambda \in (0,1)$ are fixed constants. These cylinders all have the same "vertex". Therefore, to verify that they are nested it suffices to verify that

$$\rho_{n+1}^2 \omega_{n+1}^{\frac{1-m}{m}} < \rho_n^2 \omega_n^{\frac{1-m}{m}}.$$

By making use of the definitions of ρ_n and ω_n, this is verified if (6.3) holds.

Proposition 6.1. *Let u be a weak solution of (1.1) with $\beta(\cdot)$ of the type of porous media (i.e., the first of (**ii**)). Then there exist constants $\delta, \eta \in (0,1)$ that can be determined a priori only in terms of the data, such that for every $(x_o, t_o) \in \Omega_T$,*

$$\operatorname*{ess\,osc}_{\{(x_o,t_o)+Q_{\rho_n}(\omega_n)\}} u \le \omega_n \qquad n = 0, 1, 2, \dots. \tag{6.5}$$

Here $C > 1$ and $\lambda \in (0,1)$ are numbers determined only in terms of the integrability conditions (1.5)–(1.7) and are independent of δ and η. As a consequence u is locally Hölder continuous in Ω_T.

Remark 6.1. The Hölder modulus of continuity can be derived as in the proof of Proposition 4.1, since the "shrinking" numbers δ and η are independent of the solution.[16]

Remark 6.2. It is natural to ask whether the same idea of working with intrinsically rescaled cylinders could be used for graphs of the Stefan-type. In such a case $\beta'(\cdot)$ is the Dirac mass at the origin. As a consequence the time should be intrinsically rescaled into another which, loosely speaking, would remain constant on the transition set $[u = 0]$.[17] We do not know of a general technical way of operating such a rescaling. However in [20] we have devised a variant of it, in the context of the boundary regularity of weak solutions of (1.1).

[16] The same idea of introducing an intrinsic geometry, can be applied to doubly non linear parabolic equations, as long as the singularities and/or degeneracies are power-like. We refer to Ivanov [31,32], Porzio-Vespri [46] and Vespri [54] for the main points of the theory.

[17] Presumably, a technical implementation of this idea, if at all possible, would require some preliminary information on the relative size of the singular set $[u = 0]$.

6.1 Remarks on Boundary Regularity

Suppose that *continuous* Dirichlet data are prescribed on the lateral part

$$S_T \equiv \bigcup\nolimits_{t\in(0,T]} \partial\Omega \times \{t\},$$

of the parabolic boundary of Ω_T. The boundary data on S_T are taken in the sense of the traces of the functions $u(\cdot,t) \in W^{1,1}(\Omega)$. In such a case we establish in [20] that weak solutions of (1.1) are continuous up to S_T, both for Stefan-type graphs and for graphs of the type of porous media. We introduce a time scale which becomes progressively small as as the essential oscillation of u decreases to zero. The method however could be implemented only because of the information contained in the boundary data.

7 Parabolic Equations with Multiple Singularities

Equations with $\beta(\cdot)$ exhibiting *multiple* singularities arise naturally from the flows of two immiscible fluids in a porous medium. The model example is (3.1) which, as indicated in §3, presents difficulties of similar nature as (1.1). The first attempt to establish the local continuity of solutions of (3.1) is in [1] under some assumption on the nature of the degeneracy of the function $\varphi(\cdot)$ introduced in (3.3), near *at least one* of the degeneracy points $v = 0$ and $v = 1$. For example $\varphi(\cdot)$ could degenerate at *any unrestricted rate* near $v = 1$, provided near $v = 0$ it degenerates no faster than logarithmically. The result was improved in [21] by allowing the second degeneracy to be power-like, with no restriction on the power. This last work employs a "one sided" intrinsic geometry, of the type discussed in §6, by introducing, roughly speaking, two parabolic scales. When working near the unrestricted degeneracy, say for example $v = 1$, we employ the standard parabolic cylinders $\{(x_o, t_o) + Q_\rho\}$, as in (4.1). This is because, due to the lack of information on the nature of the degeneracy, no natural rescaling is available. When working near $v = 0$, if the degeneracy is power-like, we work with cylinders coaxial with $\{(x_o, t_o) + Q_\rho\}$, with the same "vertex" at (x_o, t_o) and whose time scale is of the form (6.2).

Because on the restriction placed on the degeneracy of $\varphi(\cdot)$, both contributions [1,21] leave open the main issue of local continuity of solutions, as outlined in §3. The restriction imposed in [1,21], are used to exploit the parabolic nature of (1.1) on some side of a point of singularity of $\beta(\cdot)$.

However for *general* graphs $\beta(\cdot)$, the equation in (1.1) is not uniformly parabolic on either side of a singular point. For this reason, any continuity result for weak solutions of (1.1) with *general* $\beta(\cdot)$, would require a "non-parabolic" approach.

The first approach in this direction appears in [25], where the role played by (1.1) is reduced essentially to some energy estimates and a major role is played instead, by some novel measure-theoretical facts.[18] The results of [25] are optimal in space-dimension $N = 2$. For $N \geq 3$ they are still not complete. We will state these results and point out the main open problems regarding $N \geq 3$.

Theorem 7.1 ($N = 2$). *Let u be a locally bounded weak solution to (1.1), in the sense of (1.8)–(1.11), where $\beta(\cdot)$ is any maximal monotone graph satisfying the coercivity and boundedness conditions (1.2)–(1.3). Assume moreover that $N = 2$ and that the structure conditions (1.4)–(1.7) are satisfied for $N = 2$. Then u is locally continuous in Ω_T. Moreover, for every compact subset $\mathcal{K} \subset \Omega_T$, there exists a continuous, non-negative, increasing function*

$$s \longrightarrow \omega_{\text{data}}(s), \qquad \omega_{\text{data}}(0) = 0, \tag{7.1}$$

that can be determined a priori only in terms of the data and the distance from $\mathcal{K}$ to the parabolic boundary of Ω_T, such that

$$|u(x_1, t_1) - u(x_2, t_2)| \leq \omega_{\text{data}} \left(|x_1 - x_2| + |t_1 - t_2|^{\frac{1}{2}} \right), \tag{7.2}$$

for every pair of points $(x_i, t_i) \in \mathcal{K}$, $i = 1, 2$.

Remark 7.1. The result is optimal in that no restrictions are placed on the singularities of $\beta(\cdot)$, and the parabolic equations is permitted to bear the full quasilinear structure (1.1)–(1.7). For $N \geq 3$ on the other hand, while no restrictions are placed on $\beta(\cdot)$, the p.d.e. in (1.1) is required to have a limited structure.

Theorem 7.1 ($N \geq 3$). *Let u be a locally bounded weak solution to (1.1′), where $\beta(\cdot)$ is any maximal monotone graph satisfying the coercivity and boundedness conditions (1.2)–(1.3). Then u is locally continuous in Ω_T, with a modulus of continuity that can be determined quantitatively, a priori only in terms of the data as in (7.1)–(7.2).*

8 Main Ideas of the Proof

In outlining the main points of the proof, we let u be a weak solution of (1.1) with the full quasilinear structure (1.1)–(1.7) in any number of dimensions, and will point out later the differences between $N = 2$ and $N \geq 3$. To establish Proposition 5.1, we fix $(x_o, t_o) \in \Omega_T$ and assume, after a translation, that it coincides with the origin. We will work with the cubes K_ρ and the cylinders Q_ρ introduced in (4.1). The numbers $\mu^\pm$ and ω are defined as in Section 4.1.

Proposition 8.1. *Let $\delta \in \left(0, \frac{1}{4}\right)$ be a parameter to be chosen and assume that there exists a time level $\tilde{t} \in (-\rho^2, -\delta^2\rho^2)$, such that*

$$u\left(x, \tilde{t}\right) \leq \mu^+ - \tfrac{1}{4}\omega, \qquad \forall x \in K_{2\delta\rho}. \tag{8.1$^+$}$$

[18] The main one these is stated in Section 11 and is independent of partial differential equations. For this reason we feel that it might be applicable to other branches of Analysis.

Then there exist numbers $\eta \in (0,1)$ and $C > 1$, $\lambda \in (0,1)$, depending upon the data and δ, but independent of ω and ρ, such that either $\omega \le C\rho^\lambda$ or,

$$u(x,t) \le \mu^+ - \eta\omega \qquad \forall (x,t) \in Q_{\delta\rho} \equiv K_{\delta\rho} \times \left(-\delta^2\rho^2, 0\right). \tag{8.2$^+$}$$

Likewise if for some $\tilde{t} \in (-\rho^2, -\delta^2\rho^2)$, there holds,

$$u\left(x,\tilde{t}\right) \ge \mu^- + \tfrac{1}{4}\omega, \qquad \forall x \in K_{2\delta\rho}, \tag{8.1$^-$}$$

then either $\omega \le C\rho^\lambda$, or

$$u(x,t) \ge \mu^- + \eta\omega, \qquad \forall (x,t) \in Q_{\delta\rho} \equiv K_{\delta\rho} \times \left(-\delta^2\rho^2, 0\right), \tag{8.2$^-$}$$

for the same constants η, C, λ.

The constants $C > 1$ and $\lambda \in (0,1)$ depend only on the various parameters appearing in the structure conditions (1.5)–(1.7) and are independent of ω and the singularities of $\beta(\cdot)$. From now on we will consider them fixed.

As indicated in the proof of Proposition 4.2, either one of (8.2$^+$), (8.2$^-$) implies that going down from Q_ρ to the smaller cylinder $Q_{\delta\rho}$, the oscillation of u decreases of a factor $(1-\eta)$.

The proof of Proposition 8.1 hinges upon recursive inequalities based on the logarithmic estimates introduced in [18]. Due to the "initial conditions" (8.1$^+$), (8.1$^-$) these logarithmic estimates are analogous to those one would derive for solutions of non-singular equations.[19] Another feature of Proposition 8.1 is that the number η depends upon δ but not upon the oscillation ω. This is precisely the parameter dependence of Proposition 4.1.

Thus, the starting point of the proof is that if one had some information, such as (8.1$^+$), (8.1$^-$) on the status of the system at some "initial" time $t = \tilde{t}$, then the p.d.e. in (1.1) would behave like a quasilinear non-singular parabolic equation.

To achieve an information of the type (8.1$^+$), (8.1$^-$) we consider cylinders, coaxial with Q_ρ, with "vertex" at $(0,\tilde{t})$ and congruent to $Q_{4\delta\rho}$, i.e.,

$$\left\{(0,\tilde{t}) + Q_{4\delta\rho}\right\} \equiv K_{4\delta\rho} \times \left\{\tilde{t} - (4\delta\rho)^2\ ,\ \tilde{t}\right\}.$$

As the time level $\tilde{t}$ ranges over

$$\left\{-\left(1 - 16\delta^2\right)\rho^2\ ,\ -16\delta^2\,\rho^2\right\}, \tag{8.3}$$

the cylinders $\{(0,\tilde{t}) + Q_{4\delta\rho}\}$, move inside Q_ρ remaining coaxial with it. By moving them in the indicated range, we seek to locate some position of $\tilde{t}$ where one could derive some "initial" information of the type of (8.1$^+$), (8.1$^-$). Precisely, we will look for those positions of $\tilde{t}$, where the subset of $\{(0,\tilde{t}) + Q_{4\delta\rho}\}$ where u is

[19] The proof of Proposition 8.1 results from combining the logarithmic estimates of §4 of [25] with Propositions 3.2$^\pm$. We refer to [25] for full proofs.

close either to μ^+ or μ^- is small, i.e., for which either one of the following two inequalities holds,

$$\begin{aligned}
&\operatorname{meas}\left\{(x,t)\in\{(0,\tilde t)+Q_{4\delta\rho}\}\;\middle|\; u(x,t)\ge\mu^+-\tfrac12\omega\right\}\le\nu\,|Q_{4\delta\rho}|\,;\\
&\operatorname{meas}\left\{(x,t)\in\{(0,\tilde t)+Q_{4\delta\rho}\}\;\middle|\; u(x,t)\ge\mu^-+\tfrac12\omega\right\}\le\nu\,|Q_{4\delta\rho}|\,,
\end{aligned}\qquad(8.4^{\pm})$$

for some $\nu\in(0,1)$ to be determined in terms of the data.

Proposition 8.2. *There exists a number $\nu\in(0,1)$, that can be determined a priori only in terms of the data and ω, such that if (8.4$^+$) holds for some $\tilde t$ in the range (8.3), then either $\omega\le C\rho^\lambda$ or,*

$$u(x,t)\le\mu^+-\tfrac14\omega\qquad\forall(x,t)\in\{(0,\tilde t)+Q_{2\delta\rho}\}.\qquad(8.5^+)$$

Analogously, if (8.4$^-$) holds for some $\tilde t$, then either $\omega\le C\rho^\lambda$ or,

$$u(x,t)\ge\mu^-+\tfrac14\omega\qquad\forall(x,t)\in\{(0,\tilde t)+Q_{2\delta\rho}\}.\qquad(8.5^-)$$

The proof is based on iterative inequalities starting from energy estimates, similar to those one would obtain for quasilinear, *non-singular* equations. The singularity of $\beta(\cdot)$ contributes to these energy estimates with a large constant depending upon the data and ω. For this reason the number ν, in (8.4$^\pm$) has to be chosen to depend upon ω.[20]

A consequence of Proposition 8.2 is that if either one of (8.4$^\pm$) is verified for *some* time level $\tilde t$ in the indicated range, then at least one of (8.2$^+$), (8.2$^-$) would hold true, and the proof could be concluded as indicated in Proposition 4.2.

Therefore the unfavorable case is when both (8.4$^\pm$) are violated for *every* time level $\tilde t$ in the range (8.3). The parameter δ introduced in Proposition 8.1 is still to be chosen. We will choose it in such a way that if the unfavorable case occurs for all $\tilde t$ in the range (8.3) and for *arbitrarily small* valued of δ, then this would imply a contradiction.

Consider any one of the cylinders $\{(0,\tilde t)+Q_{4\delta\rho}\}$. If (8.4$^\pm$) are both violated for arbitrarily small δ, then near the axis of Q_ρ, at the time level $\tilde t$, there is a relatively large set where the solution u is close to μ and another relatively large set where u is close to μ^-. Since δ is arbitrarily small and $\tilde t$ is arbitrary in the range (8.3), these two sets are arbitrarily close to each other. Therefore the space gradient ∇u must be large on a relatively large set. Since however $\nabla u\in L^2(Q_\rho)$, this would create a contradiction.

9 Identifying Regions of Concentration of the Energy

The technical implementation of this idea requires that we locate those regions within $\{(0,\tilde t)+Q_{4\delta\rho}\}$ where the energy is sufficiently large. For this we identify

[20] Proposition 8.2 corresponds to Propositions $3.1^\pm$ of [25] to which we refer for a full proof and further details.

two sub-cylinders

$$\{(y_i,\tilde{t})+Q_{\delta^2\rho}\}\subset\{(0,\tilde{t})+Q_{4\delta\rho}\},\quad i=1,2, \tag{9.1}$$

where

$$\begin{aligned} u(x,t)&\geq\tfrac{1}{4}\mu^+ &\quad \forall(x,t)\in\{(x_1,\tilde{t})+Q_{\delta^2\rho}\};\\ u(x,t)&\leq\tfrac{1}{4}\mu^- &\quad \forall(x,t)\in\{(x_2,\tilde{t})+Q_{\delta^2\rho}\}.\end{aligned} \tag{9.2}$$

At first these two cylinders are found within $\{(0,\tilde{t})+Q_{4\delta\rho}\}$. Then by using the arbitrariness of $\tilde{t}$ we identify them has having their "vertices" at $(y_i,\tilde{t})$, i.e., at the same time level $\tilde{t}$. Also, using the arbitrariness of δ we may insure that their cross sections are mutually separated by a distance of at least $\delta^2\rho$.[21]

It is in this process that the new Lemma on measure theory plays a role. Assume for the moment that (8.4^+) is violated so that the set where u is close to μ^+ is relatively large. The Lemma asserts that u must be close to μ^+ in some sufficiently small box within $\{(0,\tilde{t})+Q_{4\delta\rho}\}$, i.e., the set where u is close to μ^+, even though it might be scattered in $\{(0,\tilde{t})+Q_{4\delta\rho}\}$, it must have, loosely speaking, some *concentration regions* within it. We will state and discuss this Lemma in Section 11. Here we observe that (9.2) implies

$$\tfrac{1}{4}\omega\leq u(x_1,t)-u(x_2,t)\quad\forall x_i\in\left\{y_i+K_{\delta^2\rho}\right\},\quad i=1,2, \tag{9.3}$$

for all the time levels

$$t\in\left(\tilde{t}-\delta^4\rho^2,\tilde{t}\right). \tag{9.3$'$}$$

For t fixed in the indicated range, we first integrate (9.3) over a path, piecewise parallel to the coordinate axes and joining

$$x_1\in\left\{y_1+K_{\delta^2\rho}\right\}\quad\text{ and }\quad x_2\in\left\{y_2+K_{\delta^2\rho}\right\}.$$

Then using the arbitrariness of these points within their ranges, we integrate the resulting segment-integrals, over the remaining $(N-1)$ variables, and then over the time in the range (9.3′). This yields

$$\gamma(\omega)\ (\delta\rho)^N\leq\int_{\tilde{t}-\delta^2\rho^2}^{\tilde{t}}\int_{K_{\delta\rho}\setminus K_{\delta^2\rho}}|\nabla u|^2dxd\tau, \tag{9.4}$$

where $\gamma(\omega)$ is a constant depending upon the data and ω. This inequality has been derived for all $\tilde{t}$ in the range (8.3) for which ($8.4^{\pm}$) are *both* violated. We observe that, for $\tilde{t}$ in such a range, the number of disjoint cylinders of the type $\{(0,\tilde{t})+Q_{4\delta\rho}\}$ is of the order of δ^{-2}. Thus adding (9.4) over the corresponding boxes, gives

$$\gamma(\omega)\,\delta^{N-2}\rho^N\leq\int_{-\rho^2}^{0}\int_{K_{\delta\rho}\setminus K_{\delta^2\rho}}|\nabla u|^2dxd\tau, \tag{9.4$'$}$$

[21] The proof of these assertions is in §5–8 of [25].

The argument can now be repeated with δ replaced by δ^2, since δ can be chosen to be arbitrarily small. Therefore we conclude that for all $n \in \mathbb{N}$,

$$\gamma(\omega)\delta^{n(N-2)}\rho^N \le \int_{-\rho^2}^0 \int_{K_{\delta^n\rho}\setminus K_{\delta^{n+1}\rho}} |\nabla u|^2 dx d\tau, \tag{9.4_n}$$

On the other hand, a standard energy estimate, gives

$$\iint_{Q_\rho} |\nabla u|^2 dx d\tau \le (\text{const})\rho^N. \tag{9.5}$$

We seek to derive a contradiction by iterating and adding (9.4_n) and comparing the resulting integral with (9.5).[22]

9.1 The case $N = 2$

Adding (9.4_n) for $n = 1, 2, \dots, n_o$, and taking into account (9.5) implies that,

$$\gamma(\omega)\, n_o \le (\text{const}).$$

This is a contradiction if n_o is sufficiently large depending on the data and ω. It follows that at least one of ($8.4^{\pm}$) must hold for $\tilde{t}$ in the range (8.3) and for some radius $\rho_o \in [\rho, \delta^{n_o}\rho]$. In view of Propositions 8.2 and 8.1 this would imply the result.

Remark 9.1. The same argument could be applied whenever one has information that essentially reduce the space dimension N to 1 or 2. This for example would occur for radial solutions of (1.1).

10 The Case $N \ge 3$

The key observation here is that, even though the previous argument fails if $N > 2$, an information of the type of (9.4) continues to hold within any sub-cylinder of Q_ρ not necessarily coaxial with it. With the number δ to be chosen, we assume, without loss of generality that $(4\delta)^{-1}$ is an integer, say for example m, and partition the original cube K_ρ, up to a set of measure zero, into m^N pairwise disjoint sub-cubes of wedge $(8\delta\rho)$ and centered at points $x_\ell \in K_\rho$, i.e.,

$$\begin{aligned}
&\{x_\ell + K_{4\delta\rho}\} \subset K_\rho, \qquad && \ell = 1, 2, \dots, m^N;\\
&\{x_\ell + K_{4\delta\rho}\} \cap \{x_j + K_{4\delta\rho}\} = \emptyset \quad && \text{if } \ell \ne j;\\
&K_\rho = \bigcup_{\ell=1}^{m^N} \{x_\ell + K_{4\delta\rho}\}.
\end{aligned}$$

[22] These inequalities are proved in Sections 9–12 of [25].

Then we partition the original cylinder Q_ρ, up to a set of measure zero, into m^{N+2} pairwise disjoint sub-cylinders with vertices at (x_ℓ, t_h) and all congruent to $Q_{4\delta\rho}$, i.e.,

$$\begin{aligned}
&\{(x_\ell, t_h) + Q_{4\delta\rho}\} \qquad\qquad \ell = 1, 2, \dots, m^N; h = 1, 2, \dots, m^2;\\
&\{(x_\ell, t_h) + Q_{4\delta\rho}\} \cap \{(x_j, t_k) + Q_{4\delta\rho}\} = \emptyset \text{ if } \ell \neq j \text{ or } h \neq k;\\
&Q_\rho = \bigcup_{\ell=1}^{m^N} \bigcup_{h=1}^{m^2} \{(x_\ell, t_h) + Q_{4\delta\rho}\}.
\end{aligned} \tag{10.1}$$

Returning to (8.4$^\pm$), we claim that at least one of them must be satisfied, for at least one of the cylinders $\{(x_\ell, t_h) + Q_{4\delta\rho}\}$ making up the partition of Q_ρ. Indeed if *both* of (8.4$^\pm$) are violated for *all* these cylinders, then inequality (9.4) must hold for all of them. We rewrite such inequalities in a sightly different form, i.e.,

$$\gamma(\omega)\ (\delta\rho)^N \leq \int_{t_h - \delta^2\rho^2}^{t_h} \int_{\{x_\ell + K_{\delta\rho}\}} |\nabla u|^2 dx d\tau, \qquad \begin{aligned} &\ell = 1, 2, \dots, m^N;\\ &h = 1, 2, \dots, m^2. \end{aligned}$$

Adding these inequalities over the indicated indices and taking into account (9.5) gives,

$$\gamma(\omega)\, m^2 (m\delta)^N \leq (\text{const}) \quad \Longrightarrow \quad \delta^{-2} \leq \gamma_{\text{data}}(\omega).$$

This is a contradiction for δ sufficiently small, depending on ω. It follows that at least one of (8.4$^\pm$) must hold for at least one of the cylinders (10.1) making up the partition of Q_ρ. Suppose for example that (8.4$^-$) holds true for the cylinder $\{(x_\ell, t_h) + Q_{4\delta\rho}\}$. Then, by Proposition 8.2,

$$u(x,t) \geq \mu^- + \tfrac{1}{4}\omega \qquad \forall (x,t) \in \{(x_\ell, t_h) + Q_{2\delta\rho}\}. \tag{$8.5^-_{(\ell,h)}$}$$

If $x_\ell \equiv 0$, then the cylinder $\{(x_\ell, t_h) + Q_{4\delta\rho}\}$ would be coaxial with Q_ρ, and the proof could be concluded as indicated in Proposition 8.1. Thus the main point of the proof for $N \geq 3$ is to establish that a version of ($8.5^-_{(\ell,h)}$) actually holds for a cylinder coaxial with Q_ρ. Alternatively we seek to establish that some bound below for u, within a region, would yield a bound below in a larger region. Estimates of this kind are typical of solutions of quasilinear parabolic equations and are contained for example in [44,38,52]. The difficulty here is the presence of the singularity of $\beta(\cdot)$.

In our proof such *space propagation of a bound below*, is technically realized by means of a suitable comparison function. To construct such a comparison function as well as to make full use of the comparison principle, the p.d.e. in (1.1) is required to have the restricted structure (1.1′).

10.1 Open Problems

We omit here the presentation of such a construction as we feel that the *space extension of positivity* should hold for equations with the full quasilinear structure (1.4) and it should be independent of the comparison principle. What seems to be missing is some sort of weak form of the Harnack inequality ([44,52]), for solutions of *singular* parabolic equations.

We feel that an understanding of this point would permit one also to establish a, still missing, regularity Theorem up to the parabolic boundary of Ω_T.

11 A Lemma of Measure Theory

For $r > 0$ let K_r be a cube of wedge $2r$ and centered at the origin, as in (4.1). Let $v \in W^{1,p}(K_r)$, $p > 1$ satisfy

$$\int_{K_r} |\nabla v|^p dx \le \gamma\, r^{N-p}. \tag{11.1}$$

Inequalities of this type are satisfied by harmonic functions in a domain Ω containing K_r, or more generally by solutions of quasilinear elliptic equations in divergence form.

Lemma 11.1. *Suppose that for some $\alpha \in (0,1)$ there holds*

$$\text{meas}\left\{x \in K_r \;\middle|\; v(x) < 1\right\} \ge \alpha |K_r|. \tag{11.2}$$

Then for every $\varepsilon \in (0,1)$ and $\theta > 1$, there exists some $x^* \in K_r$ and a number $\delta \in (0,1)$ that can be determined a priori only in terms of N, α, ε, θ, such that

$$\text{meas}\left\{x \in \{x^* + K_{\delta\, r}\} \;\middle|\; v(x) < \theta\right\} \ge (1-\varepsilon)|K_{\delta r}|. \tag{11.3}$$

If v were continuous in K_r, then by the Theorem of the permanence of positivity, the Lemma would be trivial. However a function $v \in W^{1,p}(K_r)$, has some regularity. Thus the Lemma can be regarded as some sort of permanence of positivity for functions in $W^{1,p}(K_r)$. It asserts that if the set where $(v-1)$ is negative, is quantitatively non negligible, then the set where $(v-\theta)$ is negative, mighty be partly scattered within K_r, provided some of it is concentrated within a full cube $\{x^* + K_{\delta\, r}\}$.

11.1 An Open Question

The proof is independent of (1.1) or any partial differential equations and makes only use of measure-theoretical arguments, starting from (11.1). The number δ deteriorates as either $\varepsilon \searrow 0$ or $\theta \searrow 1$.

The proof also uses in an essential way that $v \in W^{1,p}(K_r)$ for $p > 1$. It would be of interest to investigate it when $p = 1$.

11.2 Use of the Lemma in the Context of (1.1)

The Lemma is applied to a solution u of (1.1) in the following manner. Suppose for example that (8.4$^-$) is violated for some $\tilde{t}$ in the range (8.3). Then for some time level

$$t \in (\tilde{t} - 16r^2, \tilde{t}), \qquad \text{where} \quad r = \delta\rho, \tag{11.4}$$

there holds,

$$\operatorname{meas}\left\{x \in K_r \,\middle|\, u(x,t) > \mu^- + \tfrac{1}{2}\omega\right\} > \nu|K_r|. \tag{11.5}$$

By setting

$$v(x,t) = \frac{2\,\{u(x,t) - \mu^-\}}{\omega},$$

we rewrite (11.5) as

$$\operatorname{meas}\left\{x \in K_r \,\middle|\, v(x,t) < 1\right\} \geq \nu|K_r|. \tag{11.2$_t$}$$

Then, by possibly modifying the positive number ν into a new quantifiable positive number $\alpha \in (0,\nu)$, we establish the existence of a time τ in the range (11.4) such that the following two inequalities both hold,

$$\begin{aligned}
&\operatorname{meas}\left\{x \in K_r \,\middle|\, v(x,t) < 1\right\} \geq \alpha|K_r|,\\
&\int_{K_r} |\nabla v(x,\tau)|^2 dx \leq \gamma_{\mathrm{data}}(\omega)\, r^{N-2},
\end{aligned}$$

for a constant $\gamma_{\mathrm{data}}(\omega)$ depending only upon the data and ω, and independent of τ. Therefore by Lemma 11.1, having fixed $\varepsilon \in (0,1)$ and $\theta = \frac{3}{2}$ there exists a number $\delta \in (0,1)$ and a cube $\{x^* + K_{\delta r}\} \subset K_r$, such that

$$\operatorname{meas}\left\{x \in \{x^* + K_{\delta r}\} \,\middle|\, u(x,\tau) < \mu^- + \frac{1+\sigma}{2}\,\omega\right\} \geq (1-\varepsilon)|K_{\delta r}|.$$

It follows that one has

$$\begin{aligned}
u(x,\tau) &\leq \mu^- + \tfrac{3}{4}\omega\\
&= \mu^+ - \mu^+ + \tfrac{3}{4}(\mu^+ - \mu^-)\\
&= \mu^+ - \tfrac{1}{4}\omega,
\end{aligned} \tag{11.6}$$

everywhere in $\{x^* + K_{\delta r}\}$ except at most a set of measure less than $\varepsilon|K_{\delta r}|$. The information in (11.6) is similar to (8.1^+) where some information is available at some "initial" time $\tilde{t}$. Here the time is τ and the information is not as complete since out of $\{x^* + K_{\delta r}\}$ one has to remove a set of measure less than $\varepsilon|K_{\delta r}|$. However since $\varepsilon \in (0,1)$ is arbitrary, we establish that ε can be chosen so small that (11.6) is sufficient to apply a version of Proposition 8.1.

References

1. H. W. Alt and E. DiBenedetto, Nonsteady Flow of Water and Oil Through Inhomogeneous Porous Media, *Ann. Scuola Norm. Sup. Pisa*, Classe di Sc. Ser. IV, **XII** (3) (1985), 335–392.
2. H. W. Alt and S. Luckhaus, Quasilinear Elliptic-Parabolic Differential Equations, *Math. Z.*, **183** (1983), 311–341.

3. S. N. Antoncev, On the Solvability of Boundary Value Problems for Degenerate Two-Phase Porous Flow Equations, *Dynamika Splosnoi Sredy Vyp.*, **10** (1972), 28–53 (Russian).
4. S. N. Antoncev, A. V. Kazhikov and V. N. Monakov, *Boundary-Value Problems in the Mechanics of Nonuniform Fluids*, Studies in Mathematics and its Applications, Amsterdam, 1990.
5. D. G. Aronson and J. Serrin, Local Behavior of Solutions of Quasilinear Parabolic Equations, *Arch. Rat. Mech. Anal.* **25** (1967), 81–123.
6. J. Bear, *Dynamics of Fluids in Porous Media*, Amer. Elsevier, New York, 1972.
7. J. Bear, *Hydraulics of Groundwater*, Mc-Graw Hill, New York, 1979.
8. A. Bossavit, A. Damlamian and M. Fremond Eds., *Free Boundary Problems: Applications and Theory*, Research Notes in Mathematics, **118, 119, 120**, London 1985.
9. L. Caffarelli and L. E. Evans, Continuity of the Temperature in the two Phase Stefan Problem, *Arch. Rat. Mech. Anal.* **81** (1983), 199–220.
10. L. Caffarelli and A. Friedman, Continuity of the Density of a Gas Flow in a Porous Medium, *Trans. Amer. Math. Soc.*
11. G. Chavent and J. Jaffré, *Mathematical Models and Finite Elements Methods for Reservoir Simulation*, North-Holland, Amsterdam, 1986.
12. R. E. Collins, *Flow of Fluids Through Porous Materials*, Reinhold, New York, 1961.
13. I. I. Danilyuk, The Stefan Problem, *Russian Math. Surveys*, **40** (5), (1985), 157–223.
14. E. DeGiorgi, Sulla Differenziabilità e l'Analiticità delle Estremali degli Integrali Multipli Regolari, *Mem. Accad. Sc. Torino,* Cl. Sc. Fis. Mat. Nat., **3** (3), (1957), 25–43.
15. J. van Duijn and Zhang Hongfei, Regularity Properties of a Doubly Degenerate Equation in Hydrology, *Comm. Part. Diff. Equ.* **13**, (1988), 261–319.
16. J. Douglas Jr., Finite Differences Methods for Two-Phase Incompressible Flow in Porous Media, *SIAM J. Numer. Anal.* **20** (1983), 681–696.
17. J. Douglas Jr., B. L. Darlow, M. F. Wheeler and R. P. Kendall, Self-Adaptive Finite Element and Finite Difference Methods for Two-Phase Immiscible Flow.
18. E. DiBenedetto, Continuity of Weak Solutions to Certain Singular Parabolic Equations, *Ann. Mat. Pura ed Appl.* (IV) **CXXX** (1982), 131–177.
19. E. DiBenedetto, Continuity of Weak Solutions to a General Porous Medium Equation, *Indiana Univ. Math. Jour.,* **32** (1) (1983), 83–118.
20. E. DiBenedetto, A Boundary Modulus of Continuity for a Class of Singular Parabolic Equations, *J. of Diff. Equ.,* **63** (3), (1986), 418–447.
21. E. DiBenedetto, The Flow of Two Immiscible Fluids Through a Porous Medium: Regularity of the Saturation, *Theory and Appl. of Liquid Crystals*, Eds. J. L. Ericksen and D. Kinderlehrer, Springer-Verlag, New York 1987, 123–141.
22. E. DiBenedetto, *Degenerate Parabolic Equations*, Springer Verlag, New York 1993.
23. E. DiBenedetto and A. Friedman, Hölder Estimates for Nonlinear Degenerate Parabolic Systems, *Jour. für die Reine und Angew. Math.* **357**, (1985), 1–22.
24. E. DiBenedetto and R. Gariepy, On the Local Behavior of Solutions for an Elliptic-Parabolic Equation, *Arch. Rat. Mech. Anal.* **97** (1) (1987), 1–17.
25. E. DiBenedetto and V. Vespri, On the Singular Equation $\beta(u)_t = \Delta u$, *Arch. Rat. Mech. Anal.* **132** (1995), 247–309.
26. A. Fasano and M. Primicerio Eds., *Free Boundary Problems: Theory and Applications*, Research Notes in Mathematics, **78, 79**, Pitman, London 1983.
27. A. Friedman, *Variational Principles and Free Boundary Problems*, Wiley-Interscience, New York, 1982.

28. J. Glimm, D. Marchesin and O. McBryan, Unstable Fingers in Two Phase Flow, *Comm. Pure Appl. Math.* **34** (1981), 53–76.
29. D. Hoff, A Scheme for Computing Solutions and Interface Curves for a Doubly Degenerate Parabolic Equation, *SIAM J. Numer. Anal.* **22** (4), (1985), 687–712.
30. K. H. Hoffmann and J. Sprekels Eds., Free Boundary Problems: Theory and Applications, Pitman Research Notes in Mathematics, **185, 186**, Longman Scientific & Technical, New York, 1990.
31. A. V. Ivanov, Uniform Hölder Estimates for Generalized Solutions of Quasilinear Parabolic Equations Admitting a Double Degeneracy, *Algebra Anal.* **3** (2), (1991), 139–179.
32. A. V. Ivanov, The Classes $\mathcal{B}_{m\ell}$ and Hölder Estimates for Quasilinear Doubly Degenerate Parabolic Equations, *Zap. Nauchn. Sem. St. Petersburg Otdel. Math. Inst. Steklov (LOMI)* **197** (1991), 3–28.
33. D. Kröner and S. Luckhaus, Flow of Oil and Water in a Porous Medium *J. Diff. Equ.,* **55** (1984), 276–288.
34. S. N. Kruzkov, On the a Priori Estimation of Solutions of Linear Parabolic Equations and of Solutions of Boundary Value Problems for a Certain Class of Quasi-Linear Parabolic Equations, *Dokl. Akad. NAUK SSSR* **138** (1961), 1005–1008; (Engl. Transl. in Soviet Math. Doklady **2** (1961), 764–767).
35. S. N. Kruzkov, A Priori Estimates and Certain Properties of the Solutions of Elliptic and Parabolic Equations of Second Order, *Mat. Sbornik* **65** (107), (1968), 522–570; (Engl. Transl. Amer. Math. Soc. Transl. **2** (68), (1968), 169–220).
36. S. N. Kruzkov, Results Concerning the Nature of the Continuity of Solutions of Parabolic Equations and Some of Their Applications, *Math. Zametki* **6**, (1969), 97–108 (Russian).
37. S. N. Kruzkov and S. M. Sukorjanski, Boundary Value Problems for Systems of Equations of two Phase Porous Flow Type: Statement of the Problems, Questions of Solvability, Justification of Approximate Methods, *Mat. Sbornik,* **44** (1977), 62–80.
38. N. V. Krylov and M. V. Safonov, A Certain Property of Solutions of Parabolic Equations with Measurable Coefficients, *Math. USSR Izvestija* **16** (1) (1981), 151–164.
39. O. A. Ladyzenskaja, V. S. Solonnikov and N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Amer. Math. Soc., Providence RI, 1968.
40. M. C. Leverett, Capillary Behavior in Porous Solids, *Trans. Amer. Inst. Mining and Metallurgical Engrs.,* **142** (1941), 151–169.
41. J. L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites non Linéaires*, Dunod, Gauthiers-Villars, Paris, 1969.
42. A. Meirmanov, *The Stefan Problem*, Walter de Gruyter, Berlin, 1992.
43. J. Moser, A New Proof of DeGiorgi's Theorem Concerning the Regularity Problem for Elliptic Differential Equations, *Comm. Pure Appl. Math.* **13**, (1960), 457–468.
44. J. Moser, A Harnack Inequality for Parabolic Differential Equations, *Comm. Pure Appl. Math.* **17**, (1964), 101–134.
45. W. M. Ni, L. Peletier and J. L. Vazquez, Eds., *Degenerate Diffusion*, The IMA Volumes in Mathematics and its Appl. **47**, Springer-Verlag, New York, 1993.
46. M. M. Porzio and V. Vespri, Hölder Estimates for Local Solutions of Some Doubly Nonlinear Degenerate Parabolic Equations, *J. Diff. Equ.* **103**, (1993), 146-178.
47. P. Sachs, Continuity of Solutions of Singular Parabolic Equations, *Nonlinear Anal. TMA,* **7** (1983), 387–409.
48. P. Sachs, The Initial and Boundary Value Problem for a Class of Degenerate Parabolic Equations, *Comm. Part. Diff. Equ.* **8** (1983), 693–734.

49. A. E. Scheidegger, *The Physics of Flow Through Porous Media*, 3rd Ed., Univ. of Toronto Press, Toronto, Ontario, 1974.
50. A. Spivak, H. S. Price and A. Settari, Solution of the Equations for Multidimensional Two-Phase, Immiscible Flow by Variational Methods, *Soc. Petr. Eng. J.,* Feb. 1977, 27–41.
51. G. Stampacchia, *Equations Elliptiques du Second Ordre à Coefficients Discontinues*, Sem. Mth. Sup. **16**, Les Presses de l'Univ. de Montréal, Montréal, (1966).
52. N. S. Trudinger, Pointwise Estimates and Quasilinear Parabolic Equations, *Comm. Pure Appl. Math.* **21** (1968), 205–226.
53. V. Vespri, L^∞ Estimates for Non-Linear Parabolic Equations with natural Growth Conditions, *Rend. Sem. Mat. Univ. Padova,* **90** (1993), 1–8.
54. V. Vespri, On the Local Behavior of a Certain Class of Doubly non Linear Parabolic Equations, *Manuscripta Math.* **75**, (1992), 65–80.
55. W. P. Ziemer, Interior and Boundary Continuity of Weak Solutions of Degenerate Parabolic Equations, *Trans. Amer. Math. Soc.* **271** (1982), 773–748.