EQUADIFF 2

František Neuman

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ON BOUNDED SOLUTIONS OF A CERTAIN DIFFERENTIAL EQUATION

F. NEUMAN, Brno

We shall deal with the second-order linear differential equation

(q) $$y'' = q(t)\, y$$

where the function $q(t)$ is a continuous function on the interval $(-\infty, \infty)$, and periodic with period π. The well-known Floquet theory gives all possible types of behaviour of solutions of this differential equation. For the second-order differential equation (q) the characteristic equation

(1) $$\lambda^2 - \Delta\lambda + 1 = 0$$

is of special interest. Here, the coefficient (so called discriminant) $\Delta = u'(\pi) + v(\pi)$, where u and v are solutions of (q) determined by the initial conditions $u(0) = v'(0) = 0$, $u'(0) = v(0) = 1$. The following cases may occur:

1) If $|\Delta| > 2$ then no non-trivial solution of (q) is bounded on $(-\infty, \infty)$.
2) If $|\Delta| < 2$ then every solution of (q) is bounded on $(-\infty, \infty)$ In this case 2) the differential equation (q) is called stable.
3) If $|\Delta| = 2$ then either all solutions of (q) are bounded on $(-\infty, \infty)$ or a solution of (q) is bounded on $(-\infty, \infty)$ and every solution independent of it is unbounded on $(-\infty, \infty)$.

First, let us deal with case 2). In this case we can obtain the general solution of the differential equation (q) and at the same time the necessary and sufficient condition establishing all the stable differential equations (q). Let us restrict ourselves on the theorems only:

Theorem 1. *In every stable differential equation (q) there is the function $q(t)$ of the form*

$$q(t) = -\{\operatorname{tg} \alpha, t\}$$

where $\{\operatorname{tg} \alpha, t\}$ is Schwarz's derivative, i.e. $\frac{1}{2}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)' - \frac{1}{4}\left(\frac{\alpha''(t)}{\alpha'(t)}\right)^2 + \alpha'^2(t)$

and $\alpha(t) = P(t) + (2n + a)\,t$ *where* n *is integer,* a *is a number in the interval* $(0, 1)$, $P(t)$ *is a periodic function with period* π *such that it has continuous derivatives up to and including the order* 3 *and* $P'(t) + 2n + a \neq 0$. *The function* $P(t)$, *number* a *and integer* n *are uniquely determined by the stable differential equation* (q).

Moreover, every differential equation $y'' = q(t)\,y$ *with the function* $q(t)$ *constructed in this way is a stable differential equation* (q).

Theorem 2. *The general solution of the stable differential equation* (q) (i.e. *with* $|\Delta| < 2$) *is of the form*

$$y(t; k_1, k_2) = k_1 \frac{\sin [P(t) + (2n + a)\,t + k_2]}{\sqrt{|P'(t) + 2n + a|}}$$

where $a \in (0, 1)$ *and* $e^{\pm a\pi i}$ *are roots of the characteristic equation* (1), n *is a suitable integer and* $P(t)$ *satisfies the above conditions.*

Let us note that the integer n gives the density of zeros of the solutions.

Now, let us deal with case 3) (i.e. $|\Delta| = 2$). We I shall introduce the necessary and sufficient condition under which we may state whether all the solutions of a given differential equation (q) are bounded or not. This condition is based on the behaviour of one bounded solution of equation (q) which, in this case 3), must exist.

Theorem 3. *Let* $y(t)$ *be a non-trivial bounded solution of equation* (q) *(with* $|\Delta| = 2$*). Let* $a_1 < \ldots < a_n$ *be all zeros of* $y(t)$ *on* $[0, \pi)$. *Set*

$$r(t) = \begin{cases} \displaystyle\sum_{i=1}^{n} \frac{1}{y'^2(a_i) \sin^2 (t - a_i)} \\ 0 \text{ if there is no zero of } y(t). \end{cases}$$

Then every solution of (q) *is bounded on* $(-\infty, \infty)$ *if and only if*

$$\int_0^{\pi} \left[\frac{1}{y^2(t)} - r(t)\right] dt = 0.$$

By means of this result we may construct all the differential equations of a prescribed type. Especially, we may obtain all such second-order linear differential equations every solution of which has zeros in the same distances π. The well-known representative of such differential equations is the equation $y'' = -y$. One result of this sort:

There are $2^{\aleph_0}$ differential equations of this property. Another result:

The set of all differential equations (q) with $\Delta = -2$ all solutions of which are bounded on $(-\infty, \infty)$ and such that a solution has just one zero in the interval $[0, \pi)$ is exactly the same set as the set of all second-order linear differential equations every solution of which has zeros in the same distances equal to π.

This is the simplest result of several results establishing a close relation between disposition of zeros of solutions of a differential equation (q) and the boundedness of these solutions.