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SOME APPLICATIONS OF THE SECOND METHOD OF LIAPUNOV TO DYNAMICAL SYSTEMS DESCRIBED BY PARTIAL DIFFERENTIAL EQUATIONS

P. C. PARKS, Coventry

Abstract

The second method of Liapunov is applied to the stability of dynamical systems described by partial differential equations. This extension of the well-known technique for ordinary differential equations is illustrated by two examples drawn from the field of aeroelasticity — the torsional divergence of a wing and the supersonic flutter of a panel. Reference is made to the work of other authors working in various promising fields of application.

Introduction

When applying the second method of Liapunov to stability problems of ordinary differential equations we generally wish to show that the Euclidean state space norm $\bar{S} \equiv (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2}$ tends to zero as time tends to infinity, and the stability definitions, Liapunov theorems and their proofs are expressed in terms of $\bar{S}$ — for example in KALMAN and BERTRAM [1]. When considering systems of partial differential equations it may be possible to choose a new norm, which will involve an integral of the system dependent variables and their space and time derivatives, which provides a measure of the disturbed system, from its undisturbed state. We may then take over all the definitions and theorems for ordinary differential equations, replacing Liapunov functions by "Liapunov functionals".

This idea has been put forward by ZUBOV [2], VOLKOV [3] and MOVCHAN [4] in the U.S.S.R., but only recently have applications of this theory been seen, PARKS [5] and WANG [6].

Basic Theorems

The following theorems were given by MOVCHAN [4], but are expressed here in the language of KALMAN and BERTRAM [1].

Stability Theorem

Suppose there exists a Liapunov functional V such that when $\bar{\varrho} \neq 0$, $0 < \alpha(\bar{\varrho}) \leq V \leq \beta(\bar{\varrho})$, $V = 0$ when $\bar{\varrho} = 0$, where $\alpha(\bar{\varrho})$ and $\beta(\bar{\varrho})$ are continuous non-decreasing scalar functions of ϱ, and that $\frac{\mathrm{d}V}{\mathrm{d}t}$, making use of the partial diff. equation and its boundary conditions, is such that $\frac{\mathrm{d}V}{\mathrm{d}t} \leq -\gamma(\bar{\varrho}) < 0$, $\bar{\varrho} \neq 0$, $\gamma(\bar{\varrho}) = 0$, $\bar{\varrho} = 0$, then the system is asymptotically stable. If $\alpha(\bar{\varrho}) \to \infty$ as $\bar{\varrho} \to \infty$ then the system is asymptotically stable in the large.

Instability Theorem

Suppose there exists a Liapunov functional V such that V is bounded above in terms of $\bar{\varrho}$ and that where $V > 0$, $\frac{\mathrm{d}V}{\mathrm{d}t}$ is also positive. Suppose further that given δ however small there always exists an initial motion at time t_0 with $\bar{\varrho}(t_0) < \delta$ such that at this time $V > 0$ then the undisturbed motion is unstable.

The stability theorem is stated in a general way and provides conditions for uniform asymptotic stability. Certain relaxations may be possible, for example when considering autonomous systems.

Applications

The important aeronautical engineering field known as „aeroelasticity" provides some interesting examples of the Liapunov functional technique.

(1) Torsional Divergence of a wing

For simplicity let us consider a uniform wing (Fig. 1) in torsion under the influence of aerodynamic loads which depend on the local incidence Θ and local angular velocity $\frac{\partial \Theta}{\partial t}$.

The equation of motion for a strip element will yield

$$(1) \qquad I \frac{\partial^2 \Theta}{\partial t^2} - \frac{\partial}{\partial y}\left(GJ \frac{\partial \Theta}{\partial y}\right) = k_\Theta \Theta + k_{\dot{\Theta}} \frac{\partial \Theta}{\partial t}$$

where k_Θ and $k_{\dot{\Theta}}$ are the aerodynamic strip „derivatives".

Consider now a norm $\bar{\varrho} = \left\{ \int_0^l \Theta^2 + \left(\frac{\partial\Theta}{\partial t}\right)^2 \mathrm{d}y \right\}^{1/2}$ and a tentative Liapunov functional

$$V = \frac{1}{2} \int_0^l GJ \left(\frac{\partial\Theta}{\partial y}\right)^2 + I \left(\frac{\partial\Theta}{\partial t}\right)^2 - k_\Theta \Theta^2 \, \mathrm{d}y \tag{2}$$

for which, on substituting for $I \dfrac{\partial^2\Theta}{\partial t^2}$ from (1),

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} = \int_0^l GJ \frac{\partial\Theta}{\partial y} \frac{\partial^2\Theta}{\partial t\, \partial y} + \frac{\partial\Theta}{\partial t} \left(\frac{\partial}{\partial y} \left[GJ \frac{\partial\Theta}{\partial y} \right] + k_\Theta \Theta + k_{\dot\Theta} \frac{\partial\Theta}{\partial t} \right) - \\ - k_\Theta \Theta \frac{\partial\Theta}{\partial t} \, \mathrm{d}y = \int_0^l k_{\dot\Theta} \left(\frac{\partial\Theta}{\partial t}\right)^2 \mathrm{d}y \end{aligned} \tag{3}$$

on integrating the second term by parts and using the boundary conditions, which are that $\Theta = 0$ at $y = 0$ and $\dfrac{\partial\Theta}{\partial y} = 0$ at $y = l$.

Now $k_{\dot\Theta}$ will be negative and so we have stability (but not, without further argument, asymptotic stability) if V is positive definite in terms of $\bar{\varrho}$. Using the Schwarz inequality that

$$\left\{ \int_a^b fg \, \mathrm{d}x \right\}^2 \leq \int_a^b f^2 \, \mathrm{d}x \int_a^b g^2 \, \mathrm{d}x \tag{4}$$

we have

$$\left\{ \int_0^y \left(\frac{\partial\Theta}{\partial y}\right)^2 \mathrm{d}y \right\}^2 \leq \int_0^y \left(\frac{\partial\Theta}{\partial y}\right)^2 \mathrm{d}y \int_0^y \mathrm{d}y$$

or

$$\Theta^2(y) \leq \left[\int_0^y \left(\frac{\partial\Theta}{\partial y}\right)^2 \mathrm{d}y \right] y \leq \left[\int_0^l \left(\frac{\partial\Theta}{\partial y}\right)^2 \mathrm{d}y \right] y$$

and so

$$\int_0^l \Theta^2(y) \, \mathrm{d}y \leq \left[\int_0^l \left(\frac{\partial\Theta}{\partial y}\right)^2 \mathrm{d}y \right] \int_0^l y \, \mathrm{d}y = \left[\int_0^l \left(\frac{\partial\Theta}{\partial y}\right)^2 \mathrm{d}y \right] \frac{l^2}{2} \tag{5}$$

Thus V is positive definite in terms of $\bar{\varrho}$ if

$$GJ > \frac{1}{2} l^2 k_\Theta \tag{6}$$

Now, for a uniform wing there is an exact theory of torsional divergence found by solving the equation

$$-GJ \frac{d^2\Theta}{dy^2} = k_\Theta(\Theta + \alpha) \tag{7}$$

for $\Theta(y)$ when the wing root ($y = 0$) is at incidence α. The solution is

$$\Theta(y) = \alpha(\tan pl \sin py + \cos py - 1) \tag{8}$$

where $p^2 = k_\Theta/GJ$, and torsional divergence occurs when $pl \to \pi/2$. Thus the exact criterion is

$$GJ > \frac{4}{\pi^2} l^2 k_\Theta \tag{9}$$

Galerkin energy methods may also be applied to yield for an assumed mode $\Theta(y) = y/l$

$$GJ > \frac{1}{3} l^2 k_\Theta \tag{10}$$

and for an assumed mode $\Theta(y) = 2(y/l) - (y/l)^2$

$$GJ > \frac{2}{5} l^2 k_\Theta \tag{11}$$

We notice that the Liapunov criterion is conservative while the Galerkin methods underestimate the exact torsional stiffness required to prevent divergence as

$$\frac{1}{2} > \frac{4}{\pi^2} > \frac{2}{5} > \frac{1}{3} \tag{12}$$

(2) Panel flutter

Fig. 2 shows a pin jointed two dimensional panel. The equation of motion of this panel in supersonic flow is

$$D \frac{\partial^4 z}{\partial x^4} + m \frac{\partial^2 z}{\partial t^2} - F \frac{\partial^2 z}{\partial x^2} + \varrho a_\infty \left(U \frac{\partial z}{\partial x} + \frac{\partial z}{\partial t} \right) = 0 \tag{13}$$

where „piston theory'' has been employed in calculating the aerodynamic force on a panel element. [ϱ in (13) is air density.]

Consider a norm

$$\bar{\varrho} = \left\{ \int_{x=0}^{l} z^2 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial^2 z}{\partial x^2}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2 dx \right\}^{1/2}$$

and a tentative Liapunov functional

$$V_1 = \frac{1}{2} \int_0^l m\left(\frac{\partial z}{\partial t}\right)^2 + F\left(\frac{\partial z}{\partial x}\right)^2 + D\left(\frac{\partial^2 z}{\partial x^2}\right)^2 \mathrm{d}x \tag{14}$$

for which

$$\frac{\mathrm{d}V_1}{\mathrm{d}t} = -\int_0^l \varrho a_\infty \left(\frac{\partial z}{\partial t}\right)^2 - \varrho a_\infty U \frac{\partial z}{\partial t}\frac{\partial z}{\partial x} \mathrm{d}x \tag{15}$$

We should like a term in $\left(\frac{\partial z}{\partial x}\right)^2$ in $\frac{\mathrm{d}V}{\partial t}$ so we try

$$V = V_1 + \lambda V_2 \tag{16}$$

where

$$V_2 = \frac{1}{2} \int_0 \varrho a_\infty z^2 + 2mz\frac{\partial z}{\partial t} \mathrm{d}x \tag{17}$$

with

$$\frac{\mathrm{d}V_2}{\mathrm{d}t} = \int_0^l m \left(\frac{\partial z}{\partial t}\right)^2 - F\left(\frac{\partial z}{\partial x}\right)^2 - D\left(\frac{\partial^2 z}{\partial x^2}\right)^2 \mathrm{d}x \tag{18}$$

Now using a lemma due to Lord Rayleigh, employed also by Movchan [4] that

$$\int_0^l \left(\frac{\partial^2 z}{\partial x^2}\right)^2 \mathrm{d}x \geq \frac{\pi^2}{l^2} \int_0^l \left(\frac{\partial z}{\partial x}\right)^2 \mathrm{d}x \geq \frac{\pi^4}{l^4} \int_0^l z^2\, \mathrm{d}x \tag{19}$$

we shall have a positive definite V and $-\frac{\mathrm{d}V}{\mathrm{d}t}$ if

$$\begin{bmatrix} \frac{\pi^4 D}{l^4} + \frac{\pi^2}{l^2} F + \lambda \varrho a_\infty & \lambda m \\ \lambda m & m \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \left(\frac{\pi^2 D}{l^2} + F\right)\lambda & \frac{1}{2}\varrho a_\infty U \\ \frac{1}{2}\varrho a_\infty U & \varrho a_\infty - \lambda m \end{bmatrix}$$

are positive definite matrices. An optimum choice of λ is $\lambda = \frac{1}{2}\frac{\varrho a_\infty}{m}$ when we obtain conditions

$$\left\{ \begin{aligned} & F > -\frac{\pi^2 D}{l^2} \\ \text{and} \quad & U^2 < \left(F + \frac{\pi^2 D}{l^2}\right)/m \end{aligned} \right. \tag{20}$$

The first condition is precisely the Euler buckling criterion for the panel and the second condition, for long panels under tension, says that the air speed must be less than the speed of waves travelling in the stretched panel: this is a well known criterion, but obtained here by an unconventional method.

(3) Other aeroelasticity problems

We note the non-linear structural damping treated by PARKS [5], and the bending torsion flutter of a non-uniform wing considered by WANG [6] (but note the comments by PARKS [7]), and the body bending-tail flutter of WANG [8]. Most of these papers look at old problems using the new Liapunov technique.

(4) Other fields of applications

We note papers on a chemical reactor problem by BLODGETT [9], on plasma stability by McNAMARA and ROWLANDS [10], on instabilities in Shid dynamies by PRITCHARD [11], and on stability in elastic bodies by SHIELD [12].

There is an urgent need for further research into the construction of Liapunov functionals for these problems — physical quantities such as total energy are useful and other functionals may be generated by multiplying the differential equations through by suitable dependent variables and integrating by parts.

There are likely to be important advances in these directions before long.

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