EQUADIFF 2

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ON THE EXISTENCE AND REGULARITY OF SOLUTIONS OF NON-LINEAR ELLIPTIC EQUATIONS

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Introduction. We shall consider boundary value problems for elliptic equations of order $2k$ in the divergent form

$$\sum_{|i| \leq k} (-1)^{|i|} D^i[a_i(x, D^j u)] = f(x)$$

where D^i is the well-known symbol for derivatives in Euclidean space $E_N : D^i = \partial^{|i|}/\partial x_1^{i_1} \ldots \partial x_N^{i_N}$. We shall deal with the problem of existence of weak solutions using direct variational methods and for them the regularity theorems will be derived. In the conclusion the converse process will be used for investigation of existence of regular solution.

Contents: §1 Weak solution of the boundary value problem. Its determinig by the variational method.
§2 Regularity of the solution; application of differences method.
§3 Regularity of the solution; on the Hölder continuity of k-th derivatives.
§4 The existence of regular solution. Application of the first differential.

§1. Weak solution of the boundary value problem. Its determining by the variational method.

Let Ω be a bounded domain in E_N with Lipschitzian boundary $\partial\Omega$. Let us denote by $E(\overline{\Omega})$ the space of such real-valued infinitelly differentiable functions on Ω that can be continuously extended (with all their derivatives) to the closure of $\Omega : \overline{\Omega}$. $D(\Omega)$ is a subspace of $E(\overline{\Omega})$ which contains all functions with compact support.

Let $k \geq 1$ be an integer, $1 \leq m < \infty$. Let $W_m^{(k)}(\Omega)$ be a normed space of all real-valued functions which are integrable with m-th power over Ω and so do all their derivatives (in the sense of distributions) up to the k-th order. The norm of u is $||u||_{W_m^{(k)}} \equiv (\int_\Omega \sum_{|\alpha|\leq k} |D^\alpha u(x)|^m \, dx)^{1/m}$. Let us denote $\overset{0}{W}{}_m^{(k)}(\Omega) = \overline{D(\Omega)}$.

Let $C^{(k)}(\overline{\Omega})$ be a space of all real-valued functions which are continuous with all their derivatives up to k-th order on $\overline{\Omega}$ with usual norm and let $C^{(k),\mu}(\overline{\Omega})$ be subspace of $C^{(k)}(\overline{\Omega})$ of these functions whose k-th derivatives are μ-Hölder continuous.

We shall define functions $a_i(x, \zeta_j)$, $|i| \leq k$ for $x \in \Omega$, $-\infty < \zeta_j < \infty$, $|j| \leq k$ continuous in variables ζ_j for almost every x and measurable as functions of x for ζ_j being fixed. Each positive constant will be denoted by C. To distinguished the constants, if it is necessary we shall use indices. Let us assume

$$(1.1) \qquad |a_i(x, \zeta_j)| \leq C(1 + \sum_{|j|\leq k} |\zeta_j|^{m-1}), \qquad 1 < m < \infty$$

or less: we set $\frac{1}{q_{|i|}} = \frac{1}{m} - \frac{k-|i|}{N}$ if $(k - |i|)\, m < N$, $\frac{1}{q_{|i|}} = 0$ if $(k - |i|)\, m > N$, $\frac{1}{q_{|i|}} > 0$ if $(k - |i|)\, m = N$. For $1 \leq q \leq \infty$ let $q' = \frac{q}{q-1}$, $\varkappa_{|i|,|j|} = \frac{q_{|j|}}{q'_{|i|}}$ and let $C(s)$ be continuous non-negative function for $0 \leq s < \infty$. Let $g_i \in L_{q'_{|i|}}(\Omega)$, $g_{|i|}(x) \geq 0$. Let us suppose

$$(1.2) \qquad |a_i(x, \zeta_j)| \leq C\,(\sum_{(j)<k-N/m} |\zeta_j|)\, (g_i(x) + \sum_{k-N/m\leq|j|\leq k} |\zeta_j|^{\varkappa_{|i|,|j|}}).$$

The following assertion is valid: the operator $a_i(x, D^j u)$ is continuous from $W_m^{(k)}(\Omega)$ into $L_{q'_{|i|}}(\Omega)$. Its proof is based upon imbedding theorems for $W_m^{(k)}(\Omega)$ spaces. (See, for instance, E. Cagliardo [10] and also M. M. Vajnberg [28].)

Let now be $D(\Omega) \subset \mathfrak{Y} \subset E(\Omega)$, $V = \overline{\mathfrak{Y}}$ in $W_m^{(k)}(\Omega)$ and let Q be such Banach space that $D(\overline{\Omega}) = Q$ and that $\overset{0}{W}{}_m^{(k)}(\Omega) \subset Q$ algebraically and topologically. Let $u_0 \in W_m^{(k)}(\Omega)$ (stable boundary condition), $g \in V'$ such functional that $gv = 0$ for $v \in \overset{0}{W}{}_m^{(k)}(\Omega)$ (unstable boundary condition), and $f \in Q'$ (the right-hand side) be given. Let us denote $gv = \langle v, g\rangle_{\partial\Omega}$, $fv = \langle v, f\rangle_\Omega$.

Definition of the boundary value problem and of weak solution: We are looking for such $u \in W_m^{(k)}(\Omega)$ that

$$(1.3) \qquad u - u_0 \in \overset{0}{W}{}_m^{(k)}(\Omega),$$

$$(1.4) \qquad \text{for each } v \in V\colon \int_\Omega \sum_{|i|\leq k} D^i v\, a_i(x, D^j u)\, dx = \langle v, f\rangle_\Omega + \langle v, g\rangle_{\partial\Omega}.$$

Thus, boundary value problem (1.3), (1.4), we shall transfer to the problem of finding a minimum of certain functional $\Phi(v)$. There are many other aspects the problem can be approached. Thus, many authors have dealt with the existence of the solution of boundary value problem using the concept of "monotone operators" which we shall use further. (See, e. g. F. E. Browder [2], [3], M. I. Višik [30], J. Leray, J. L. Lions [17]...) We shall obtain similar results; the difference is that we shall suppose certain additional condition concerning symmetry of the operator. But we shall know that certain functional has minimum in our solution. If the functional is a priori known then further considerations are analogic to those in papers: F. E. Browder [6], M. M. Vajnberg, R. I. Kačurovskij [29]. See also the book by S. G. Michlin [18].

The condition of symmetry: Let d be the number of indices with lenght $|i| \leq k$, $\varphi \in D(E_d)$. Then (1.5) holds almost ewerywhere in Ω:

$$(1.5) \qquad (-1)^{|j|} \int_{E_d} \frac{\partial \varphi}{\partial \zeta_j} a_i(x, \zeta_\alpha)\, \mathrm{d}\zeta = (-1)^{|i|} \int_{E_d} \frac{\partial \varphi}{\partial \zeta_i} a_j(x, \zeta)\, \mathrm{d}\zeta.$$

There is proved in author's paper [20] (using the formula for integration of differential, see M. M. Vajnberg [28]):

Theorem 1.1. *Let the conditions* (1.2) *and* (1.5) *be satisfied. Then*

$$(1.6) \qquad \Phi(v) = \int_0^1 \mathrm{d}t \int_\Omega \sum_{|i| \leq k} D^i v a_i(x, D^j u_0 + t D^j v)\, \mathrm{d}x - \langle v, f\rangle_\Omega - \langle v, g\rangle_{\partial\Omega}$$

is continuous functional on V; *its Gateaux' differential is*

$$(1.7) \qquad D\Phi(v, \tilde{v}) \equiv \lim_{\tau \to 0} \frac{\Phi(v + \tau\tilde{v}) - \Phi(v)}{\tau} = \int_\Omega \sum_{|i| \leq k} D^i \tilde{v} a_i(x, D^j u_0 + D^j v)\, \mathrm{d}x - $$
$$- \langle \tilde{v}, f\rangle_\Omega - \langle \tilde{v}, g\rangle_{\partial\Omega}.$$

To prove the existence of minimum $\Phi(v)$ on V, we shall investigate the conditions under which the following relations hold:

$$(1.8) \qquad \lim_{\|v\|\, W_m^{(k)} \to \infty} \Phi(v) = \infty$$

(1.9) $\quad \Phi(v)$ is weakly lower-semicontinuous.

If v is the point of minimum of $\Phi(v)$, then $D\Phi(v, \tilde{v}) = 0$, which is (1.4). Differential (1.7) is said to be totally monotone (strictly totally monotone) if for all $v, w \in V$, $v \neq w$,

$$(1.10) \quad \int_\Omega \sum_{|i| \leq k} D^i(w - v)\, [a_i(x, D^j u_0 + D^j w) - a_i(x, D^j u_0 + D^j v)]\, \mathrm{d}x \geq 0, \ (> 0)$$

holds.

We shall say that the differential (1.7) is coercitive if for all $v \in V$

$$(1.11)\quad \int_\Omega \sum_{|i|\le k} D^i v a_i(x, D^j u_0 + D^j v)\,dx \ge \lambda(||v||\, W_m^{(k)}) \qquad \text{holds}$$

where $\lambda(s)/s \in L_1(0, R)$ for every $R > 0$ and $\lim_{R\to\infty} \frac{1}{R} \int_0^R \frac{\lambda(s)}{s}\,ds = \infty.$

There is proved in author's paper [20].

Theorem 1.2. *Let* (1.2), (1.5), (1.10), (1.11) *be satisfied. Then there exists* min $\Phi(v)$ *(*$\Phi(v)$ *is defined by* (1.6)*), namely, in the point* v. *Function* $v + u_0$ *is the solution of problem. If the condition* (1.10) *of the strict monotony is satisfied, the solution is unique. In this case* $\Phi(v_n) \to \Phi(v) \Rightarrow v_n \rightharpoonup v$ *(weak convergence).*

Let us remark that (1.11) is satisfied, e.g. if $u_0 = 0$ and

$$\sum_{|i|\le k} \zeta_i a_i(x, \zeta_j) \ge C \sum_{|i|=k} |\zeta_i|^m + C\,.\,|\zeta_{(0,0\cdots 0)}|^m.$$

If

$$\sum_{|i|\le k} (\xi_i - \eta_i)\,.\,[a_i(x, \xi_j) - a_i(x, \eta_j)] \ge 0$$

then (1.10) is satisfied e.t.c. See author's paper [20].

Let us write the operators $a_i(x, D^j u)$ in the form $a_i(x, D^\alpha u, D^\beta u)$ where the symbol $D^\alpha u$ denotes a vector of derivatives with $|\alpha| = k$ and $D^\beta u$ a vector with $|\beta| < k$.

We say that the main part of the differential (1.7) is monotone if for v, $w, \omega \in V$

$$(1.12)\quad \int_\Omega \sum_{|i|=k} D^i(w - v)\,[a_i(x, D^\alpha u_0 + D^\alpha w, D^\beta u_0 + D^\beta \omega) - a_i(x, D^\alpha u_0 + D^\alpha v, D^\beta u_0 + D^\beta \omega)]\,dx \ge 0$$

holds.

Let us investigate the conditions under which the functional (1.6) is weakly lower-semicontinuous. For this we need monotony of the highest derivatives [see condition (1.12)] and strengthened continuity which is to be locally uniform regardig the derivatives $D^\alpha u$.

Sufficient conditions for this are following:

Let $c(s)$, $d(s)$ be continuous functions for $0 \le s < \infty$, non-negative, $d(0) = 0$ and assume

$$(1.13)\quad |i| = k : |a_i(x, \zeta_\alpha, \xi_\beta) - a_i(x, \zeta_\alpha, \eta_\beta)| \le \le c(\max(\sum_{|\beta|<k-N/m} |\xi_\beta|, \sum_{|\beta|<k-N/m} |\eta_\beta|))\cdot[d(\sum_{|\beta|<k-N/m} |\xi_\beta - \eta_\beta|)\,\cdot \cdot(1 + \sum_{|\alpha|=k} |\zeta_\alpha|^{m-1}) + \sum_{|\alpha|=k,k-N/m\le|\beta|<k} |\zeta_\alpha|^\lambda\, |\xi_\beta - \eta_\beta|^{\mu\,|\beta|}],$$

where $0 < \mu_{|\beta|} < q_{|\beta|}\cdot\frac{m - 1 - \lambda}{m}$. Let

$$(1.14)\quad a_i(x, \zeta_\alpha, \zeta_\beta) = \sum_{|\alpha|=k} \zeta_\alpha a_{i\alpha}(x, \zeta_\beta) + a_i(x, \zeta_\beta)$$

hold for $|i| < k$. Let $a_{i\alpha} \not\equiv 0$ at most when $q_{|i|} > \frac{m}{m-1}$. Let us suppose

$$(1.15)\quad |a_{i\alpha}(x, \zeta_\beta)| \leq c(\sum_{|\beta|<k-N/m} |\zeta_\beta|) \cdot (1 + \sum_{k-N/m \leq |\beta| < k} |\zeta_\beta|^{\nu_{|\beta|}})$$

where $0 \leq \nu_{|\beta|} < \frac{(m-1)\, q_{|i|} - m}{m \cdot q_{|i|}} \cdot q_{|\beta|}$ and

$$(1.16)\quad |a_i(x, \zeta_\beta)| \leq c\,(\sum_{|\beta|<k-N/m} |\zeta_\beta|) \cdot (g_i(x) + \sum_{k-N/m \leq |\beta| \leq k} |\zeta_\beta|^{\varkappa^*_{|i|,|\beta|}}),$$

where $g_i(x) \geq 0$, $g_i \in L_{q^*_{|i|}}$ and $q^*_{|i|} > q'_{|i|}$ if $k - N/m \leq |i|$; $q^*_{|i|} = 1$ if $|i| <$ $< k - N/m$. Further $\varkappa^*_{|i|,|\beta|} < \frac{q_{|\beta|}}{q'_{|i|}}$.

We can prove (see again [20]).

Theorem 1.3. *Let the conditions* (1.2), (1.5), (1.11), (1.12), (1.13), (1.14), (1.15), (1.16) *be satisfied. Then there exists a minimum of* (1.6); *let us denote it v. Function* $v + u_0$ *is the solution of problem.*

Let us remark, that

$$(1.17)\quad \sum_{|i|=k} (\xi_i - \eta_i) \cdot [a_i(x, \xi_\alpha, \zeta_\beta) - a_i(x, \eta_\alpha, \zeta_\beta)] \geq 0$$

is sufficient for the validity of (1.12).

§2. Regularity of the solution; application of differences method.

E. Hopf in his article [14] and many other authors have used this method to prove the regularity of solution of non-linear second order elliptic equations. Thus it is possible to obtain properties of $k + 1$-st derivatives. Author doesn't know how to apply this method, if it is possible, when investigating regularity of the derivatives of $k + 2$-nd and higher orders (as for the nonlinear elliptic equations in general form).

We shall assume, that functions $a_i(x, \zeta_j)$ are continuously differentiable for $x \in \overline{\Omega}$, $-\infty < \zeta_j < \infty$ and we denote $a_{ij}(x, \zeta_\alpha) = \frac{\partial a_i}{\partial \zeta_j}(x, \zeta_j)$. Assuming $m \geq 2$, we restrict ourselves to the following conditions (see [20]):

$$
(2.1)\quad \begin{cases}
|a_{ij}(x,\zeta_\alpha)| \le c|\zeta_i|^{\frac{m}{2}-1} \cdot |\zeta_j|^{\frac{m}{2}-1}, \quad |i| = |j| = k, \\
|a_{ij}(x,\zeta_\alpha)| \le c|\zeta_j|^{\frac{m}{2}-1} \cdot \Big(1 + \sum_{|\alpha|\le k} |\zeta_\alpha|^{\frac{m}{2}-1}\Big), \ |i| < k, \ |j| = k; \\
\text{analogically for } |i| = k, \ |j| < k, \\
|a_{ij}(x,\zeta_\alpha)| \le c\,.\,(1 + \sum_{|\alpha|\le k} |\zeta_\alpha|^{m-2}), \quad |i| < k, \ |j| < k; \\
\sum_{|i|=|j|=k} a_{ij}(x,\zeta_\alpha)\,\xi_i\xi_j \ge c \sum_{|i|=k} |\zeta_i|^{m-2}\xi_i^2, \\
\left|\frac{\partial a_i}{\partial x_l}(x,\zeta_\alpha)\right| \le c\,.\,|\zeta_i|^{\frac{m}{2}-1}\Big(1 + \sum_{|\alpha|\le k} |\zeta_\alpha|^{\frac{m}{2}}\Big) \ \text{for } |i| = k, \\
\left|\frac{\partial a_i}{\partial x_l}(x,\zeta_\alpha)\right| \le c(1 + \sum_{|\alpha|\le k} |\zeta_\alpha|^{m-1})
\end{cases}
$$

or to the conditions

$$
(2.2)\quad \begin{cases}
|a_{ij}(x,\zeta_\alpha)| \le c(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{2}-1}, \quad |i| = |j| = k, \ d \ge 0, \\
|a_{ij}(x,\zeta_\alpha)| \le c(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{2}} \,.\, (1 + \sum_{|\alpha|\le k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{2}}, \quad |i| = k, \ |j| < k \\
\text{and analogically for } |i| < k, \ |j| = k, \\
|a_{ij}(x,\zeta_\alpha)| \le c\,(1 + \sum_{|\alpha|\le k} \zeta_\alpha^2)^{\frac{m}{2}-1} \ \text{for } |i| < k, \ |j| < k, \\
c_1(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{2}-1}|\xi|^2 \le \sum_{|i|=|j|=k} a_{ij}(x,\zeta_\alpha)\,\xi_i\xi_j \le c_2(d + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{2}-1}\,|\xi|^2, \\
\left|\frac{\partial a_i}{\partial x_l}\right| \le c\,(1 + \sum_{|\alpha|\le k} \zeta_\alpha^2)^{\frac{m}{2}-\frac{1}{2}}, \quad |i| < k, \\
\left|\frac{\partial a_i}{\partial x_l}\right| \le c(1 + \sum_{|\alpha|=k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{4}}\,(1 + \sum_{|\alpha|\le k} \zeta_\alpha^2)^{\frac{m}{4}-\frac{1}{4}}, \quad |i| = k.
\end{cases}
$$

Let us denote by $\sigma(x)$ an infinitely differentiable function which is equivalent with dist $(x, \partial\Omega)$ and which satisfies $|D^i\sigma| \le c\,.\,\sigma^{1-|i|}$. (Existence of such function is proved by author in [22].)

We shall consider smoothness of the solution in Ω, not in $\overline{\Omega}$. We shall assume that the right-hand side satisfies an inequality

$$
(2.3)\quad \sum_{l=1}^{N} \left\| \frac{\partial f}{\partial x_l}\,\sigma^k \right\|_{W_2^{(-k)}(\Omega)'} \le c,
$$

where $W_2^{(-k)}(\Omega)$ is the dual space to $\overset{\circ}{W}{}_2^{(k)}(\Omega)$.

Applying the standard differences method (see e.g. J. NEČAS [21]) we obtain

Theorem 2.1. *Let $u \in W_m^{(k)}(\Omega)$, $m \geq 2$ be the solution of problem* (1.3), (1.4). *(Generally we do not suppose* (1.5).*) Let* (2.1), (2.3) *also be satisfied. Then*

$$\int_\Omega \sigma^{2k} \sum_{l=1}^{N} \sum_{|i|=k} \left(\frac{\partial}{\partial x_l} |D^i u|^{\frac{m}{2}} \right)^2 \mathrm{d}x < c$$

and thus ($N \geq 3$):

$$(2.4) \quad \int_\Omega \sum_{|i| \leq k} \sigma^{\frac{2kN}{N-2}} \, . \, |D^i u|^{\frac{mN}{N-2}} \, \mathrm{d}x \leq c < \infty,$$

$$(2.5) \quad \int_\Omega \sum_{|i| \leq k} \sigma^{2kp} |D^i u|^p \, \mathrm{d}x \leq C_p < \infty, \quad 1 < p < \infty, \; N = 2.$$

Similarly the next theorem is valid:

Theorem 2.2. *Let $u \in W_m^{(k)}(\Omega)$, $m \geq 2$ be the solution of problem* (1.3), (1.4) *(Generally we do not suppose* (1.5).*) Let* (2.2), (2.3) *also be satisfied. Then the inequalities*

$$\int_\Omega \sigma^{2k} \, . \sum_{l=1}^{N} \left(\frac{\partial}{\partial x_l} [\, d + \sum_{|\alpha|=k} (D^\alpha u)^2]^{\frac{m}{4}} \right)^2 \mathrm{d}x \leq c < \infty,$$

$$\int_\Omega \sigma^{2k} \, . \, [d + \sum_{|\alpha|=k} (D^\alpha u)^2]^{\frac{m}{2} - 1} . \sum_{|i|=k+1} (D^i u)^2 \, \mathrm{d}x \leq c < \infty$$

and (2.4), (2.5) *hold.*

Analogical assertion is valid if we set $\sum_{|\alpha| \leq k} \zeta_\alpha^2$ instead of $\sum_{|\alpha|=k} \zeta_\alpha^2$ in (2.2).

If $k = 1$ (the equation of second order) we can weaken our requirements. Let us denote functions $a_i(x, \zeta_j)$ by symbols: $a_i(x, u, p)$, $i = 1, 2, \ldots, N$) $a(x, u, p)$, where $p = (p_1, \ldots, p_N)$, $p_i = \dfrac{\partial u}{\partial x_i}$ and let $\nu(s)$, $\mu(s)$, $\mu_1(s)$ be non-negative functions for $0 \leq s < \infty$. Let us denote $|p| = (\sum_{i=1}^{N} p_i^2)^{1/2}$. Let us assume

$$(2.6) \quad \begin{cases} \nu(|u|) \, . \, (1 + |p|)^{m-2} \, . \sum_{i=1}^{N} \xi_i^2 \leq \sum_{i,j=1}^{N} \dfrac{\partial a_i}{\partial p_j}(x, u, p) \, \xi_i \xi_j \leq \\ \leq \mu(|u|) \, . \, (1 + |p|)^{m-2} \sum_{i=1}^{N} \xi_i^2, \\ \sum_{i=1}^{N} \left(\left| \dfrac{\partial a_i}{\partial u} \right| + |a_i| \right) . \, (1 + |p|) + \sum_{i,j=1}^{N} \left| \dfrac{\partial a_i}{\partial x_j} \right| + |a| \leq \\ \leq \mu(|u|) \, . \, (1 + |p|)^m, \; 1 < m < \infty \end{cases}$$

$$(2.7)\quad \begin{cases} \sum_{i,k=1}^{N} \left|\frac{\partial a_i}{\partial x_k}(x, u, p)\right| . (1 + |p|) + \sum_{i=1}^{N} \left|\frac{\partial a}{\partial p_i}(x, u, p)\right| . (1 + |p|) + \\ + \left|\frac{\partial a}{\partial u}(x, u, p)\right| + \sum_{i=1}^{N} \left|\frac{\partial a}{\partial x_i}(x, u, p)\right| \leq \mu_1(|u|) . (1 + |p|)^m, \\ 1 < m < \infty. \end{cases}$$

Let $u \in W_m^{(1)}(\Omega)$ be a weak solution satisfying the next condition: for each $\varphi \in D(\Omega)$ the equation

$$(2.8)\quad \int_\Omega \left(\sum_{i=1}^{N} a_i(x, u, p) \frac{\partial \varphi}{\partial x_i} + a(x, u, p)\, \varphi\right) \mathrm{d}x = 0$$

holds. Then the next assertion holds (see O. A. Ladyženskaja, N. N. Uralceva [16]):

Theorem 2.3. *Let* $u \in W_m^{(1)}(\Omega)$, $1 < m < \infty$ *be the weak solution satisfying* (2.8), *let* $\sup_{x \in \Omega} |u(x)| < \infty$. *Let* (2.6) *and* (2.7) *be valid. Then for* $\overline{\Omega}' \subset \Omega$

$$(2.8)'\quad \int_{\Omega'} (1 + |p|)^{m-2} \sum_{i,j=1}^{N} \left(\frac{\partial^2 u}{\partial x_i\, \partial x_j}\right)^2 \mathrm{d}x \leq c(\Omega') < \infty \text{ holds.}$$

If $k = 1$, $u_0 \in C^2(\overline{\Omega})$ and if we consider the Dirichlet problem we can substitute Ω for Ω' in Theorem 2.3 when $\partial\Omega$ in sufficiently smooth. (See [16].)

Analogical results concerning the solution of the variational problem for the functional $\int_\Omega f(x, u, p)\, \mathrm{d}x$ (as Theorem 2.3 and following) proved C. B. Morey [19]. Let $f(x, u, p)$ be a function which has two Hölder continuous derivatives according to each variable and let the inequality

$$(2.9)\quad C_1(1 + u^2 + |p|^2)^{\frac{m}{2}} - C_3 \leq f(x, u, p) \leq C_2\, (1 + u^2 + |p|^2)^{\frac{m}{2}}$$

be satisfied for $1 < m < \infty$.

Furthermore, let $u_0 \in W_m^{(1)}(\Omega)$. Let us look for such

$$(2.10)\quad u \in W_m^{(1)}(\Omega),\ u - u_0 \in \overset{0}{W}{}_m^{(1)}(\Omega),$$

that

$$(2.11)\quad \int_\Omega f\left(x, u, \frac{\partial u}{\partial x}\right) \mathrm{d}x \text{ is minimal.}$$

The solution u satisfies Euler equation in the weak form: for $\varphi \in D(\Omega)$:

$$(2.12)\quad \int_\Omega \left(\sum_{i=1}^{N} \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial f}{\partial p_i}(x, u, p) + \varphi \frac{\partial f}{\partial u}(x, u, p)\right) \mathrm{d}x = 0.$$

Let us denote $\frac{\partial f}{\partial p_i} = a_i(x, u, p), \frac{\partial f}{\partial u}(x, u, p) = a(x, u, p)$.

Let

$$(2.13)\quad \begin{cases} |a_i(x, u, p)| + \left|\dfrac{\partial a_i}{\partial x_l}(x, u, p)\right| + |a(x, u, p)| + \left|\dfrac{\partial a}{\partial x_l}(x, u, p)\right| \leq \\ \leq C(1 + u^2 + |p|^2)^{\frac{m}{2}-\frac{1}{2}}, \\ \left|\dfrac{\partial a_i}{\partial u}\right| + \left|\dfrac{\partial a}{\partial u}\right| \leq C(1 + u^2 + |p|^2)^{\frac{m}{2}-1}, \\ C_1(1 + u^2 + |p|^2)^{\frac{m}{2}-1} \sum_{i=1}^{N} \xi_i^2 \leq \sum_{i,j=1}^{N} \dfrac{\partial a_i}{\partial p_j}(x, u, p)\, \xi_i \xi_j \leq \\ \leq C_2(1 + u^2 + |p|^2)^{\frac{m}{2}-1} \sum_{i=1}^{N} \xi_i^2 \end{cases}$$

be satisfied. (Comp. with (2.2).) Then (see C. B. Morey [19]):

Theorem 2.4. *If $u \in W_m^{(1)}(\Omega)$, $m \geq 2$, u satisfies* (2.12) *and if the conditions* (2.13) *are satisfied then* (2.8)′ *holds. If $1 < m < 2$ then there exists u satisfying* (2.12) *such that* (2.8)′ *holds again.*

See also E. R. Buley [6].

§3. Regularity of the solution; on the Hölder continuity of k—th derivatives.

Under the assumptions of the Theorems 2.1 or 2.2 we have (3.1) for the weak solution and $\varphi \in D(\Omega)$

$$(3.1)\quad \int_\Omega \sum_{|i|,|j|\leq k} a_{ij}(x, D^\alpha u)\, D^i\varphi D_j \frac{\partial u}{\partial x_l}\, dx =$$
$$= -\int_\Omega \sum_{|i|\leq k} \frac{\partial a_i}{\partial x_l}(x, D^\alpha u)\, D^i\varphi\, dx + \left\langle \varphi, \frac{\partial f}{\partial x_l} \right\rangle_\Omega, \quad l = 1, 2, \ldots, N.$$

Thus if we denote $\omega = \dfrac{\partial u}{\partial x_l}$ then ω is a weak solution of linear differential equation. The investigation of regularity of higher derivatives is based upon (3.1) and upon regularity theorems for the linear equations. In this section we restrict ourselves to the assumptions (2.2) with $d = 1$. Simple example can be given to exhibit that conditions (2.1) do not guarantee continuity of $k + 1$-st derivatives in Ω in spite of the analyticity of functions $a_i(x, \zeta_j)$, $f(x)$. (See J. Nečas [20].)

If $k = 1$ then (3.1) yields further information if we set $\varphi = \dfrac{\partial u}{\partial x_l} b_n^s\, \psi^2$, $\psi \in$

$\in D(\Omega)$, $s \geq 0$, $b_n(x) = \min(|p|^2, n)$, $n = 1, 2, \ldots$ (φ — the comparison function). See e.g. O. A. LADYŽENSKAJA, N. N. URALCEVA [16]. The comparison function

$$\varphi = d_n^s \frac{\partial u}{\partial x_l} \psi^2, \quad \psi \in D(\Omega), \quad s \geq 0, \tag{3.2}$$
$$d_n = \min\{(1 + u^2 + |p|^2), n\}, \quad n = 1, 2, \ldots$$

has been used in E. R. BULEY's paper [6] under assumptions (2.9), (2.13) and $m \geq 2$. The same function has been used by C. B. MOREY [19] but with $s < 0$. From this the boundedness of the first derivatives on every $\Omega' \subset \subset \bar{\Omega}' \subset \Omega$ can be obtained when $s \to \infty$. (See E. R. BULEY [6], J. NEČAS [21].) If

$$\sup_{\Omega'} |p(x)| \leq C(\Omega') < \infty \tag{3.3}$$

is proved and if (2.8)′ holds then $\frac{\partial u}{\partial x_l} = \omega$ is a weak solution of linear equation with bounded and measurable coefficients on Ω' according to (2.1). When $k = 1$ we can use DE GIORGI's result $\left(\text{if } \frac{\partial f}{\partial x_l} = 0 \text{ see [12]}\right)$ or more general result of G. STAMPACCHIA $\left(\text{if } \frac{\partial f}{\partial x_l} \neq 0\right)$ [27]:

Theorem 3.1. *Let $u \in W_2^1(\Omega)$ be a weak solution of the equation: for $\varphi \in D(\Omega)$,*

$$\int_\Omega \sum_{i,j=1}^N a_{ij} \frac{\partial \varphi}{\partial x_i} \cdot \frac{\partial u}{\partial x_j} \,.\, dx = \int_\Omega \varphi f \, dx + \int_\Omega \sum_{i=1}^N \frac{\partial \varphi}{\partial x_i} f_i,$$

where $f \in L_p(\Omega)$, $f_i \in L_{2p}(\Omega)$, $p > \frac{N}{2}$, $a_{ij} \in L_\infty(\Omega)$, $\sum_{i,j=1}^N a_{ij}\xi_i\xi_j \geq C|\xi|^2$, then there exists such $0 < \mu < 1$ that

$$||u||_{C^{(0),\mu}(\bar{\Omega}')} \leq C(\Omega') \left(||f||_{L_p(\Omega)} + \sum_{i=1}^N ||f_i||_{L_{2p}(\Omega)} + ||u||_{W_2^{(1)}(\Omega)}\right), \quad \bar{\Omega}' \subset \Omega$$

holds.

The proof of Hölder continuity for higher derivatives and (for $k = 1$ of the analyticity of solution) follows e.g. by the result of A. DOUGLIS, L. NIRENBERG [9] (or by results of E. HOPF [14]). We shall formulate the results: E. R. BULEY [6]:

Theorem 3.2. *Let $k = 1$, $m \geq 2$, let u be the solution of* (2.10), (2.11) *and let* *the assumptions* (2.9), (2.13) *be satisfied. Then* (3.3) *holds and there exists* $0 < \mu < 1$ *that*

(3.4) $\quad ||u||_{C^{(1)\mu}(\Omega')} \leq C(\Omega') < \infty$ *holds.*

Applying C. B. MORREY's result the Theorem 3.2 can be obtained for such u which satisfies the condition (2.12). Furthermore, this author obtained:

Theorem 3.3. *Let* $k = 1$, $1 < m < 2$ *and otherwise let all assumptions of the preceding theorem be satisfied. Then there exists such solution of the problem* (2.10), (2.11), *that* (3.1), (3.4) *hold.*

O. A. LADYŽENSKAJA, N. N. URALCEVA:

Theorem 3.4. *Let* $u \in W_m^{(1)}(\Omega)$, $1 < m < \infty$ *be a weak solution which satisfies the condition* (2.8). *Let* $\sup_{x \in \Omega} |u(x)| < \infty$ *and let* (2.6), (2.7) *hold. Then* (3.3), (3.4) *hold.*

The inequality (3.3) was essential in proof of regularity of the solution for $k = 1$. The inequality (3.1) ($k = 1$) has been considered by many authors that generalized the result of T. RADO [26] under essentially weaken assumptions (supposing that $\Omega = \Omega'$, $\partial\Omega$ is smooth and Ω is strictly convex). (See e.g. P. HARTMAN, G. STAMPACCHIA [13], D. GILBARG [11].)

Now let us consider $k \geq 2$. The use of the comparison function of the type (3.2) does not lead to any result and the information

(3.5) $$\sup_{x \in \Omega'} \sum_{|i| \leq k} |D^i u(x)| \leq C(\Omega') < \infty$$

is not available. Accordingly, we shall consider the case $m = 2$ or we shall suppose that (3.5) holds. Thus we transfer the problem of regularity of k-th derivatives to the linear problem.:

Let A_{ij} be a real matrix of bounded measurable functions in a domain O, $|i| = |j| = k$. We shall use the following assumptions:

(3.6) $$C_1|\zeta|^2 \leq \sum_{|i|=|j|=k} A_{ij}\zeta_i\zeta_j \leq C_2|\zeta|^2,$$

(3.7) $$A_{ij} = A_{ji}.$$

Function $w \in W_2^{(k)}(O)$ is a weak solution of the equation $\sum_{|i|=|j|=k} D^i(A_{ij}D^j w) = \sum_{|i|=k} D^i f_i$ with $f_i \in L_2(O)$, if for each $\varphi \in D(\Omega)$

(3.7)′ $$\int_O \sum_{|i|=|j|=k} A_{ij}D^i\varphi D^j w \,\mathrm{d}x = \int_O \sum_{|i|=k} D^i\varphi f_i \,\mathrm{d}x.$$

Further let us denote $O\ d = \{x \in O, \operatorname{dist}(x, \partial O) = d\}$, $B(x_0, r) = \{x, |x - x_0| < r\}$. For $0 < \lambda < N$ let $\mathscr{L}^{(2,\lambda)}(O)$ be such subspace of $L_2(O)$ that

$$\sup_{x_0 \in O, \varrho > 0} (\varrho^{-\lambda} \cdot \int_{B(x_0, \varrho) \cap O} f^2(x)\, dx)^{1/2} \equiv ||f||_{\mathscr{L}^{(2,\lambda)}(O)} < \infty.$$

For the properties of these spaces see, e.g. S. CAMPANATO [7].

Applying S. CAMPANATO's method [7] whose generalization for the equation of higher order has been given in the paper [15] of J. KADLEC and J. NEČAS, we obtain the following:

Theorem 3.5. *Let w be a weak solution satisfying* (3.7)'. *If* (3.6), (3.7) *and if*

$$(3.8) \qquad \lambda = \frac{N \cdot \log \dfrac{1 - \frac{3}{4}\frac{C_1}{C_2}}{1 - \frac{C_1}{C_2}}}{\log \dfrac{2AC_2}{C_1} + \log \dfrac{1 - \frac{3}{4}\cdot\frac{C_1}{C_2}}{1 - \frac{C_1}{C_2}}} > N - 2,$$

$$(3.9) \qquad f_i \in \mathscr{L}^{(2,\lambda)}(O_d), \; d > 0$$

are satisfied then we obtain

$$||W||_{C^{(k-1),\mu}(\bar{O}_d)} \leq C(d) \sum_{|i|=k} ||f_i||_{\mathscr{L}^{(2,\lambda)}(O_{d/2})}, \; \mu = \frac{\lambda + 2 - N}{2}.$$

(3.8) *is always satisfied for $N = 2$. For $N \geq 3$ it holds when the positively-definite matrix $\frac{1}{C_2} \cdot A_{ij}$ is sufficiently near (uniformly on O) to the unit matrix in the sense of* (3.8). *The constant A is absolute.*

The Theorem 3.5 is — in certain sense — an analogy of the Theorem 3.1 for $k \geq 2$.

If

$$(3.10) \qquad ||A_{ij}||_{C(\bar{O})} \leq C < \infty$$

holds, then (see [15]):

Theorem 3.6. *Let w be a weak solution satisfying* (3.7)' *and let the assumptions* (3.6), (3.10), (3.9) *with $\lambda > N - 2$ be satisfied. Then*

$$||w||_{C^{(k-1),\mu}(\bar{O}_d)} \leq C(d) \sum_{|i|=k} ||f_i||_{\mathscr{L}^{(2,\lambda)}(O_{d/2})}, \; \mu = \frac{\lambda + 2 - N}{2} \text{ holds.}$$

Replace $\left\langle \varphi_i, \frac{\partial f}{\partial x_l} \right\rangle_\Omega$ in (3.1) by the expression $\int_\Omega \sum_{|i|=k} D^i\varphi \frac{\partial f_i}{\partial x_l} dx$ where

$$(3.12) \qquad \int_\Omega \sum_{|i|=k} \sum_{l=1}^{N} \left(\frac{\partial f_i}{\partial x_l}\right)^2 \sigma^{2k}\, dx < \infty.$$

Further let us suppose that (2.2) is valid and (for technical reason)

(3.13) $a_i \equiv 0$ for $|i| < k$, $\dfrac{\partial a_i}{\partial \varphi_j} \equiv 0$ for $|j| < k$, $a_{ij} = a_{ji}$.

According to Theorems 2.2, 3.5, 3.6 we obtain (see J. Nečas [23]):

Theorem 3.7. *Let $u \in W_m^{(k)}(\Omega)$, $m \geq 2$ be a solution of the problem* (1.3), (1.4) *and let the assumptions* (2.2), (3.12), (3.13) *be satisfied (the constants C_1, C_2 have the same meaning as before). Then we have*

(a) *if $m = N = 2$ and*

$$(3.14) \qquad \sum_{|i|=k} \sum_{l=1}^{N} \left\| \frac{\partial f_i}{\partial x_l} \right\|_{\mathscr{L}^{(2,\lambda)}(\Omega_d)} \leq C d^{-k}, \ d > 0$$

then $||u||_{C^{(k),\frac{\lambda}{2}}(\bar{\Omega}_d)} \leq C d^{-k-\frac{\lambda}{2}}$, *($\lambda$ is taken of* (3.8))

(b) *if $m > 2$, $N = 2$,* (3.5) *has the form* $\sup\limits_{x \in \Omega_d} \sum\limits_{|i|=k} |D^i u(x)|^2 \equiv A_d \leq C_3 d^{-\alpha}$ *and if* (3.14) *with*

$$\gamma \geq 2\mu_d \equiv \frac{2 \,.\, \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}}{\log 2A \dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}} + \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3 d^{-\alpha})^{1-\frac{m}{2}}}}$$

is valid then $||u||_{C^{(k),\mu_d}(\bar{\Omega}_d)} \leq \dfrac{C}{\mu_d} d^{-k-\mu_d}$,

(c) *if $m = 2$, $N \geq 3$,* $\dfrac{\partial a_i}{\partial x_l} = 0$, (3.8) *is valid with the constants C_1, C_2 from* (2.2) *and if* (3.14) *with λ from* (3.8) *is satisfied then*

$$||u||_{C^{(k),\frac{\lambda-N+2}{2}}(\bar{\Omega}_d)} \leq C d^{-k-\frac{\lambda}{2}},$$

(d) *if $m \geq 2$, $N \geq 3$ and* (3.5) *in the form* $\sup\limits_{x \in \Omega} \sum\limits_{|\alpha|=k} |D^\alpha u(x)|^2 \leq C_3$ *is satisfied, further if* (3.8), (3.14) *with*

$$\lambda = \frac{N \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}}{\log 2A \dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}} + \log \dfrac{1 - \dfrac{3}{4}\dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}{1 - \dfrac{C_1}{C_2}(1 + C_3)^{1-\frac{m}{2}}}}.$$

is valid then $||u||_{C^{(k),\frac{\lambda-N+2}{2}}(\bar{\Omega}_d)} \leq Cd^{-k-\frac{\lambda}{2}}$

(e) *if* $m \geq 2, N \geq 2$, $||u||_{C^{(k)}(\bar{\Omega})} \leq C_3$ *and if* (3.14) *with* $\lambda > N-2$ *is valid then*

$$||u||_{C^{(k),\frac{\lambda-N+2}{2}}(\bar{\Omega}_d)} \leq Cd^{-k-\frac{\lambda}{2}}.$$

§4. The existence of the regular solution. Application of the first differential.

Let Ω be a bounded domain with infinitely differentiable boundary $\partial\Omega$. Let $a_i(x, \zeta_j, t)$ be real functions with the same meaning as in section §1, defined for $|i| \leq k$ continuous on $\bar{\Omega}\ x(-\infty < \zeta_j < \infty)\ x(0 \leq t \leq 1)$ and continuously differentiable in ζ_j, t and let $a_i(x, 0, 0) = 0$. Using the same notation as above i.e. $a_{ij} = \frac{\partial a_i}{\partial \zeta_j}$ we suppose

$$(4.1)\quad \begin{cases} |a_{ij}(x, \zeta_\alpha, t) - a_{ij}(y, \eta_\alpha, t)| \leq \\ \leq C_2(\sum\limits_{|\alpha|\leq k} (|\zeta_\alpha| + |\eta_\alpha|)) \cdot (|x-y|^\mu + \sum\limits_{|\alpha|\leq k} |\zeta_\alpha - \eta_\alpha|) \\ \text{and the same for } \frac{\partial a_i}{\partial t}, \\ |a_{ij}(x, \zeta_\alpha, t_1) - a_{ij}(y, \eta_\alpha, t_1) + a_{ij}(y, \eta_\alpha, t_2) - a_{ij}(x, \zeta_\alpha, t_2)| \leq \\ \leq C_2(\sum\limits_{|\alpha|\leq k} |\zeta_\alpha| + |\eta_\alpha|))\, \omega(|t_1 - t_2|)(|x-y|^\mu + \sum\limits_{|\alpha|\leq k} |\zeta_\alpha - \eta_\alpha|), \end{cases}$$

and the same for $\frac{\partial a_i}{\partial t}$

where $C_2(s)$ is a non-negative continuous function for $0 \leq s < \infty$, $0 < \mu < 1$ and $\omega(s)$ is continuous function for $0 \leq s < \infty$, $\omega(0) = 0$.

Let us assume further

$$(4.2)\qquad C_1(\sum_{|\alpha|\leq k} |\eta_\alpha|)|\zeta|^2 \leq \sum_{|i|=|j|=k} a_{ij}(x, \eta_\alpha, t)\, \zeta_i \zeta_j$$

where $C_1(s)$ is a continuous positive function for $0 \leq s < \infty$. Further let $f_i \in C^{(0),\mu}(\bar{\Omega})$, $|i| \leq k$, $u_0 \in C^{(k),\mu}(\bar{\Omega})$. Let us denote by $\mathring{C}^{(k),\mu}(\bar{\Omega})$ the subspace of $C^{(k),\mu}(\bar{\Omega})$ whose elements are functions for which $\frac{\partial^l u}{\partial n^l} = 0$ on $\partial\Omega$, $l = 0, 1, \ldots k-1$. (The derivation in the direction of exterior normal.) We look for such weak solution of the Dirichlet problem $u \in C^{(k),\mu}(\bar{\Omega})$ that

$$(4.3)\qquad u - u_0 \in \mathring{C}^{(k),\mu}(\bar{\Omega})$$

$$(4.4)\qquad \text{for each } \varphi \in D(\Omega) \int_\Omega \sum_{|i|\leq k} D^i\varphi a_i(x, D^j u, 1)\, dx = \int_\Omega \sum_{|i|\leq k} D^i\varphi f_i\, dx.$$

Let the functions $b_i(x, D^j u, t)$, $|i| \leq k$, $|j| < k$ be continuous on $\bar{\Omega} \times -\infty <$

$< \zeta_j < \infty \times 0 \leq t \leq 1$ continuously differentiable in ζ_j, t, $b_i(x, 0, 0) = 0$. Let us denote $b_{ij} = \frac{\partial b_i}{\partial \zeta_j}$ and assume that b_{ij}, $\frac{\partial b_i}{\partial t}$ satisfy the conditions (4.1).

Roughly speaking, we shall solve the problem (4.3), (4.4) as follows: We shall look for such curve $u(t)$, $0 \leq t \leq 1$ with its values in $C^{(k),\mu}(\overline{\Omega})$ that $u(t)$ satisfies the problem (4.3), (4.4) with tu_0, tf_i. For this curve we shall obtain a differential equation $\frac{\mathrm{d}u}{\mathrm{d}t} = N[t, u(t)]$ and we shall look for such solution that $u(0) = 0$. See J. NEČAS [24] see also F. E. BROWDER [4]. Thus instead of solving the problem (4.3), (4.4) we look for a mapping $u(t, \tau)$ with a domain $\tau = 0$, $0 \leq t \leq 1$, $t = 1$, $0 \leq \tau \leq 1$ and a range in $C^{(k),\mu}(\overline{\Omega})$ which is continuous with its derivative $\frac{\partial u}{\partial t}(t, 0)$ from $0 \leq t \leq 1$ to $C^{(k),\mu}(\overline{\Omega})$ for $\tau = 0$. (The case when $a_i(x, \zeta_j, t)$ does not depend on t is of great importance.) Further we require

$$(4.5) \qquad u(t, \tau) - tu_0 \in \mathring{C}^{(k),\mu}(\overline{\Omega}),$$

$$(4.6) \qquad \varphi \in D(\Omega): \int_\Omega \sum_{|i| \leq k} D^i\varphi a_i(x, D^j u, t)\,\mathrm{d}x + (1 - \tau) \int_\Omega \sum_{|i| \leq k} D^i\varphi b_i(x, D^j u, t)\,\mathrm{d}x = t \int_\Omega \sum_{|i| \leq k} D^i\varphi f_i\,\mathrm{d}x.$$

Further let us assume that for $\|u\|_{C^{(k),\mu}(\overline{\Omega})} \leq R \leq \infty$ the following holds: if $w \in \mathring{W}_2^{(k)}(\Omega)$ and (4.7) holds for every $\varphi \in D(\Omega)$:

$$(4.7) \qquad \int_\Omega \sum_{|i|,|j| \leq k} a_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x + \int_\Omega \sum_{|i|,|j| \leq k} b_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x = 0$$

then $w \equiv 0$. This assumption implies the existence of only one element (for $\|u\|_{C^{(k),\mu}(\overline{\Omega})} \leq R$, *if* $R < \infty$) $w \in C^{(k),\mu}(\overline{\Omega})$ for which

$$(4.8) \qquad w - u_0 \in \mathring{C}^{(k),\mu}(\overline{\Omega})$$

$$(4.9) \qquad \text{for } \varphi \in D(\Omega): \int_\Omega \sum_{|i|,|j| \leq k} a_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x + \int_\Omega \sum_{|i|,|j| \leq k} b_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\,\mathrm{d}x = -\int_\Omega \sum_{|i| \leq k} \left(\frac{\partial a_i}{\partial t}(x, D^\alpha u, t) + \frac{\partial b_i}{\partial t}(x, D^\alpha u, t)\right) D^i\varphi\,\mathrm{d}x + \int_\Omega \sum_{|i| \leq k} f_i D^i\varphi\,\mathrm{d}x \quad \text{is valid.}$$

It follows e.g. from the article by S. AGMON, A. DOUGLIS, L. NIRENBERG [1] or from J. KADLEC, J. NEČAS [15].

Let us denote by $w = N(u, t, f_i, u_0)$ the mapping that assigns to a function $u \in C^{(k),\mu}(\overline{\Omega})$ from the sphere $\|u\|_{C^{(k),\mu}(\overline{\Omega})} \leq R$, to the parameter t from $\langle 0, 1 \rangle$, to the elements f_i, $|i| \leq k$ and to the element u_0 the function w. Now, we have for a function $w \in \overset{\circ}{C}{}^{(k),\mu}(\overline{\Omega})$, which is a weak solution of the equation

$$\int_\Omega \sum_{|i|,|j|\leq k} a_{ij}(x, D^\alpha u, t)\, D^i\varphi\, D^j w\, \mathrm{d}x + \int_\Omega \sum_{|i|,|j|\leq k} b_{ij}(x, D^\alpha u, t)\, D^i\varphi D^j w\, \mathrm{d}x = $$
$$= \int_\Omega \sum_{|i|\leq k} G_i D^i\varphi\, \mathrm{d}x$$

that there holds:

$$(4.10)\quad \|\omega\|_{C^{(k),\mu}(\overline{\Omega})} \leq C_3(\|u\|_{C^{(k),\mu}(\overline{\Omega})}, \mu) \sum_{|i|\leq k} \|G_i\|_{C^{(0),\mu}(\overline{\Omega})}$$

where $C_3(\eta_1, \eta_2)$ is continuous and positive function for $0 \leq \eta_1 < \infty$, $0 < < \eta_2 < 1$. According to this it follows:

(4.11)
- (a) The mapping $N(u, t, f_i, u_0)$ is locally Lipschitzian: for $\|u_l\|_{C^{(k),\mu}(\overline{\Omega})} \leq \leq R_0 < \infty$, $l = 1, 2$, $R_0 \leq R$, $0 \leq t \leq 1$, $\|f_i\|_{C^{(k),\mu}(\overline{\Omega})} \leq R_1 < \infty$, $\|u_0\|_{C^{(k),\mu}(\overline{\Omega})} \leq R_1$ there is $\|w_1 - w_2\|_{C^{(k),\mu}(\overline{\Omega})} \leq C(R_0, R_1) \|u_1 - u_2\|_{C^{(k),\mu}(\overline{\Omega})}$,
- (b) N is continuous as the mapping $u, t \to w$,
- (c) N is continuous in f_i, u_0 uniformly with respect to $\|u\|_{C^{(k),\mu}(\overline{\Omega})} \leq \leq R_0$, $0 \leq t \leq 1$.

For $\tau = 0$ we have: if $u(t, 0)$ is a solution of the problem (4.5), (4.6) for $0 \leq t < \varepsilon$ and if $0 < \varepsilon \leq 1$, $\|u(t, 0)\|_{C^{(k),\mu}(\overline{\Omega})} \leq R$ then

$$(4.12)\quad \frac{\partial u}{\partial t}(t, 0) = N(u(t), t, f_i, u_0),\ 0 \leq t < \varepsilon,\ u(0, 0) = 0$$

holds and thus

$$(4.13)\quad u(t, 0) = \int_0^t N(u(s), s, f_i, u_0)\, \mathrm{d}s,\ 0 \leq t < \varepsilon.$$

Now, using the standart method based upon the theorem of contraction, owing to the validity of (4.11) we obtain the existence of the solution of (4.13) for some interval $\langle 0, \varepsilon\rangle$, $\varepsilon > 0$; if there is such solution for some interval $\langle 0, \varepsilon\rangle$, $\varepsilon < 1$ then it also exists for the interval

$$\langle 0, \varepsilon_1\rangle, \qquad 1 \geq \varepsilon_1 > \varepsilon.$$

We assume that $u(t, 0)$ is such solution on the interval $\langle 0, \varepsilon\rangle$ and that

$$(4.14)\quad \|N(u(t), t, f_i, u_0)\|_{C^{(k),\mu}(\overline{\Omega})} \leq F(\|u(t)\|_{C^{(k),\mu}(\overline{\Omega})})$$

holds for $t \in \langle 0, \varepsilon\rangle$, where $F(s)$ is continuous and non-decreasing function for $s \in \langle 0, \infty)$, $F(0) > 0$. Let $y(t)$ be the solution of Cauchy problem $y(0) = 0$, $y'(t) = F(y(t))$. Evidently the following holds:

$$(4.15)\quad \|u(t)\|_{C^{(k),\mu}(\overline{\Omega})} \leq y(t).$$

But (4.15) implies the existence of the solution of (4.13) wherever $y(t)$ is defined, i. e. for $0 \leq t \leq \varepsilon$, where

$$(4.16) \qquad \varepsilon < \int_0^\infty \frac{dz}{F(z)}.$$

According to this we have

Theorem 4.1. *Let the assumptions* (4.1), (4.2), (4.7) *with* $R = \infty$ *be satisfied and let* $b_i(x, \zeta_j, t) \equiv 0$. *Then there exists a solution of the problem* (4.3), (4.4) *if* $\int_0^\infty \frac{dz}{F(z)} > 1$. *Otherwise there exists a solution of the problem* (4.5), (4.6) *for* εf_i, εu_0 *where* $\varepsilon < \int_0^\infty \frac{dz}{F(z)}$. *If an a priori estimate* $||u(t)||_{C^{(k),\mu}(\bar{\Omega})} \leq \frac{R}{2} < \infty$ *is known (u is a solution of* (4.5), (4.6)*), where R is from* (4.11) *then there exists a solution of the problem because it is possible to set* $F(z) = \text{const}$.

If there exists a function from (4.14) *with* $\int_0^\infty \frac{dz}{F(z)} > 1$ *uniformly with respect to some neighbourhood of* f_i, u_0 *then the solution* $u(1,0)$ *is continuous in* f_i, u_0 *in this neighbourhood.*

Theorem 4.2. *Let the assumptions* (4.1), (4.2) *and the following condition* (4.17) *be satisfied*:

(4.17) *If* $\sum_{|i| \leq k} ||g_i||_{C^{(0),\mu}(\bar{\Omega})} \leq C$, u_0 *being fixed*, $u(t, 0)$ *is an eventual solution for* tu_0, tg_i, *then there exists such continuous non-negative function* $R(a)$ *that* $||u(t, 0)||_{C^{(k),\mu}(\bar{\Omega})} \leq R(a)$ *and* (4.7) *holds with* $2R(a)$.

Furthermore let the "a priori" estimate $||u(1, \tau)||_{C^{(k),\mu}(\bar{\Omega})} \leq \varrho$ *hold for* u_0, f_i *being fixed. Then there exists a solution of the problem* (4.3), (4.4).

Actually, according to the preceding theorem, our problem has a solution if $\tau = 0$ (for considered u_0 and arbitrary g_i) constructed above. (It is possible to guarantee the existence of this solution also under different assumptions, see the preceding theorem.) Let $A(g_i)$ be this solution. Let us consider the mapping $A(f_i - \tau b_i(x, D^j u, 1))$ from $\langle 0, 1\rangle \times C^{(k),\mu}(\bar{\Omega})$ to $C^{(k),\mu}(\bar{\Omega})$ for $0 \leq \leq \tau \leq 1$. This mapping represents homotopy of compact transformations and the mapping $A - u$ is different from zero on the boundary of the sphere $B_{2\varrho} \equiv ||u||_{C^{(k),\mu}(\bar{\Omega})} \leq 2\varrho$. Now, for the degree of mapping with respect to O and to the sphere in question we have

$$d[A(f_i - b_i(x, D^j u, 1)) - u, 0, B_{2\varrho}] = d[A(f_i) - u, 0, B_{2,\varrho}] = -1.$$

Hence there exists the solution of our problem. See J. Cronin [8].

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