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NONLINEAR FUNCTIONAL ANALYSIS AND NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS.[1]

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Introduction: The two basic approaches to fundamentally nonlinear problems in partial differential equations are on the one hand, variational methods (the direct method of the calculus of variations, the Morse theory, and the Lusternik—Schnirelman theory) and on the other hand, the theory of nonlinear operators in Banach spaces (the Schauder fixed point theorem, the Leray—Schauder theory of the degree for compact displacements). In the past few years, we have seen a merging of these two lines of ideas in their applications to partial differential equations through the theory of monotone operators from a Banach space X to its conjugate space X^*, i.e. operators T such that for all u and v in the domain of T, we have

$$(Tu - Tv,\ u - v) \approx 0,$$

(where (w, v) denotes the pairing between the functional w and the element v). On the one hand, every operator T which is the derivative (or subderivative) of a convex functional on X is monotone, and on the other hand, the consideration of monotone (or quasi-monotone, or semi-monotone) operator equations falls within the framework of nonlinear functional analysis, i.e. the study of nonlinear operators and nonlinear operator equations in Banach spaces.

It is our object in the present paper to give a survey of some recent work by the writer on this type of functional analysis and its applications to various types of abstract differential equations in Hilbert and Banach spaces. We refer the reader to an earlier survey ([6]) for a development of the basic ideas in the application of monotone operators to such topics as:

(1) The existence of solutions for variational boundary problems for nonlinear elliptic differential operators of the form

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$$A(u) = \sum_{|\alpha| \leq m} D^\alpha(A_\alpha(x, u, \ldots, D^m u)).$$

(2) The corresponding existence theorems for parabolic operators of the form:

$$\frac{\partial u}{\partial t} + A_t(u) = 0.$$

(3) Nonlinear equations of evolution in Hilbert and Banach spaces arising from initial-boundary value problems of various types.

Section 1 below presents the results of [14] on nonlinear equations of evolution in Hilbert space and the generalized method of steepest descent for monotone operators in Hilbert space. Section 2 develops the results of the extension of this theory as carried through in [16] to Banach spaces. both for monotone operators T from a Banach space X to its dual space X^* and for J-monotone operators T from a Banach space X to X. Section 3 discusses the general method developed in [15] for proving the existence of periodic solutions for classes of nonlinear equations of evolution in infinite dimensional spaces comparable to the classes of differential equations treated in Sections 1 and 2.

We remark that the method of steepest descent and its generalizations have close links with the ideas of the calculus of variations, and the results presented below are connected with extensions of the results given in BROWDER [7] on the application of the Lusternik—Schnirelman principle to the proof of the existence of infinitely many eigenfunctions for nonlinear elliptic eigenvalue problems.

Section 1: Let H be a real Hilbert space, T an operator (generally nonlinear) with domain and range in H. We consider three inter-related problems concerning such operators T:

(I) The existence for a given w in H of solutions u of the equation $Tu = w$.

(II) The existence for a given u_0 of solutions of the nonlinear equation of evolution

$$\frac{du}{dt} = -T(u), \qquad t \geq 0,$$

with $u(0) = u_0$.

(III) For a suitably chosen perturbation term $R(t, u)$ which converges to zero as $t \to +\infty$, the convergence as $t \to +\infty$ of solutions of the equation

$$\frac{du}{dt} = -T(u) + R(t, u)$$

to solutions v_0 of the stationary equation $Tv_0 = 0$.

We denote this last problem as that of the generalized method of steepest descent for the operator T.

We recall that an operator T is said to be hemicontinuous if it is continuous from each line segment in $D(T)$ to the weak topology of H.

Theorem 1.1: *Let T be a monotone operator in the Hilbert space H such that either: (i) $D(T) = H$, and T maps H hemicontinuously into H; or (ii) $T = L + T_0$ where L is a maximal accretive closed linear operator in H and T_0 is a hemicontinuous monotone mapping of H into H which maps bounded subsets into bounded subsets.*

Suppose that there exists $R > 0$ such that for u in $D(T)$ with $||u|| = R$, $(Tu, u) \geq 0$.

Then the set of solutions u of the equation $Tu = 0$ is a nonempty closed convex subset K of H.

Theorem 1.2: *Let T be a hemicontinuous locally bounded operator from H to H such that for a fixed constant c in R^1 and all u and v of H,*

$$(Tu - Tv, u - v) \leq c||u - v||^2.$$

Then there exists one and only one strongly continuous, veakly once-differentiable function u from $R^+ = \{t| \, t \in R^1, t \geq 0\}$ to H such that u is a solution of the differential equation

$$\frac{du}{dt} = Tu, \qquad t \geq 0,$$

with the initial condition $u(0) = u_0$, for a given u_0 in H.

In addition, if T is continuous, then u is strongly C^1.

Theorem 1.3: *Let H be a Hilbert space, f a mapping of $R^+ \times H$ into H such that the following three conditions are satisfied:*

(1) *f is locally bounded (i.e. bounded on some neighborhood of each point of $R^+ \times H$). For each fixed t in R^+, $f(t, \cdot)$ is a hemicontinuous mapping of H into H. For each fixed u in H, $f(\cdot, u)$ is continuous from R^+ to the weak topology of H.*

(2) *There exists a continuous function c from R^+ to R^1 such that for all t in R^+ and all u and v in H:*

$$(f(t, u) - f(t, v), u - v) \leq c(t) \, ||u - v||^2.$$

(3) *For each u in H, $f(t, u)$ is weakly once differentiable from R^+ to H, and there exists a continuous function q from $R^+ \times R^+$ to R^+ such that for all u and t:*

$$\left\| \left(\frac{\partial}{\partial t}\right) f(t, u) \right\| \leq q(t, ||u||).$$

Then for any u_0 in H, there exists one and only one function u from R^+ to H

which is weakly continuously once-differentiable and which satisfies the differential equation

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$.

Theorems (1.2) and (1.3) are sharpenings (under more restrictive hypotheses on the dependence of f on t) of an existence theorem given in BROWDER [3] with the additional assumption that $f(t, u)$ is bounded for t and u ranging through a bounded set of $R^+ \times H$. The interest of this strenghthening lies primarily in the fact that it is obtained through a new a priori estimate for solutions of these equations of evolution from which one obtains much stronger control over the solutions of these equations. This is brought out more clearly in the following theorems on nonlinear evolution equations containing an unbounded linear operator L.

Definition: *Let H be a Hilbert space, $\{L(t) | t \in R^+\}$ a family of closed, densely defined linear operators in H, T_0 a mapping of $R^+ \times H$ into H. If we set $T_t(u) = L(t)\, u + T_0(t, u)$, then by a sharp solution u on R^+ of the equation of evolution*

$$\frac{du}{dt} = T_t(u), \qquad t \geq 0,$$

we mean a strongly continuous function u from R^+ to H with u weakly once continuously differentiable from R^+ to H, $u(t)$ in the domain of $L(t)$ for each t in R^+ and with $L(t)\, u(t)$ weakly continuous from R^+ to H, and such that for all t in R^+,

$$\frac{du}{dt}(t) = L(t)\, u(t) + T_0(t, u(t)).$$

Theorem 1.4: *Let H be a Hilbert space, L a maximal dissipative linear operator in H, T_0 a mapping of $R^+ \times H$ into H which maps bounded sets into bounded sets. Suppose that T_0 satisfies the following three conditions:*

(1) *For each fixed t in R^+, $f(t, \cdot)$ is a hemicontinuous mapping of H into H. For each fixed u in H, $f(\cdot, u)$ is continuous from R^+ to the weak topology of H.*

(2) *There exists a continuous function c from R^+ to R^1 such that for all t in R^+ and all u and v in H:*

$$(T_0(t, u) - T_0(t, v), u - v) \leq c(t)\, ||u - v||^2.$$

(3) *For each fixed u in H, $f(t, u)$ is weakly once-differentiable on R^+ in t, and there exists a continuous function q from $R^+ \times R^+$ to R^+ such that for all t in R^+ and all u in H,*

$$\left|\left|\left(\frac{\partial}{\partial t} T_0\right)(t, u)\right|\right| \leq q(t, ||u||).$$

Then for each u_0 in $D(L)$, there exists one and only sharp solution u on R^+ of the equation of evolution

$$\frac{du}{dt} = Lu + T_0(t, u), \qquad t \geq 0.$$

with $u(0) = u_0$.

As an illustration of the basic a priori bounds from which these results are derived, we have the following:

Theorem 1.5: *Let L and T_0 satisfy the conditions of Theorem* **1.4** *and let u be a sharp solution of the differential equation*

$$\frac{du}{dt} = Lu + T_0(t, u).$$

Let $C(t) = \int_0^t c(s)\, ds$. *Then:*

(I) *For all t in R^+,*

$$||u(t)|| \leq \exp\,(C(t))\, ||u(0)|| + \int_0^t \exp\,(C(t) - C(s))\, ||T_0(s, 0)||\, ds.$$

(II) *If $q(t, r)$ is nondecreasing in r (as we may always assume) and if* $||u(s)|| \leq$ $\leq M(s)$ *for all s in R^+, then*

$$\left|\left|\frac{du}{dt}(t)\right|\right| \leq \exp\,(C(t))\, ||T_0(0, u(0)) + Lu(0)|| + \int_0^t \exp\,(C(t) - C(s))\, q(s, M(s))\, ds.$$

Combining these apriori estimates with the corresponding existence theorems, we obtain the following general result on the generalized method of steepest descent for monotone operators in Hilbert spaces:

Theorem 1.6: *Let H be a Hilbert space, T a monotone operator with domain in H and values in H which lies in one of the two following classes:*

(*a*) *T is a locally bounded hemicontinuous mapping of H into H.*

(*b*) *$T = L + T_0$, where L is a maximal accretive linear operator in H, and T_0 is a hemicontinuous monotone mapping of H into H which carries bounded subsets into bounded subsets.*

Suppose that there exists $R > 0$ such that $(Tu, u) \geq 0$ for all u in $D(T)$ with $||u|| = R$.

Let c be a C^1 function from R^+ to R^+ which is non-increasing and such that $c(t) \to 0$ as $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$.

Let v_0 be any element of H with $||v_0|| < R$, u_0 any element of $D(L)$ with $||u_0|| \leq R$.

Then:

(1) *The equation of evolution*

$$\frac{du}{dt} = -T(u) - c(t)\{u - v_0\}, \qquad t \geq 0,$$

has one and only one sharp solution u *on* R^+ *with* $u(0) = u_0$.

(2) *As* $t \to +\infty$, *this solution converges strongly in* H *to a solution* w_0 *of the equation* $Tw = 0$. *This limit is characterized as that solution of* $Tw = 0$ *in the ball* $B_R = \{u \mid ||u|| \leq R\}$ *closest to the given element* v_0.

Section 2: We now turn to the generalizations and extensions of the results of Section 1 to more general Banach spaces than Hilbert space, as given by the writer in BROWDER [16]. These extensions are of two kinds:

(1) The consideration of monotone operators T from X to X^*.

(2) The consideration of J-monotone operators T from X to X, for a duality mapping J of X into X^*.

We shall consider case (1) first.

Definition: *Let* X *be a Banach space, with* $X \subset H \subset X^*$ *for a Hilbert space* H, *in the sense that we are given continuous linear injections of each space on a dense subset of its successor and the pairing between two elements* w *and* u *of* H *with* w *in* X *and* u *in* X^* *coincides with the* H *inner product.*

Let f *be a mapping of* $R^+ \times X$ *into* X^*.

Then a function u *from* R^+ *to* X *is said to be a sharp solution on* R^+ *of the equation of evolution*

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0$$

if u *satisfies the following three conditions:*

(1) u *is continuous from* R^+ *to the weak topology of* X.

(2) *As a function from* R^+ *to* H, u *is continuous to the strong topology of* H *and satisfies a Lipschitz condition in* H *on each finite interval.* u *is strongly once-differentiable in* H *a.e. on* R^+ *and* $\left\|\frac{du}{dt}(t)\right\|_H$ *is essentially bounded on each finite interval.*

(3) *The differential equation*

$$\frac{du}{dt}(t) = f(t, u(t))$$

holds a.e. on R^+.

To abbreviate these hypotheses, we use the following notation: If Y is a Banach space, $C_s^0(R^+, Y)$ and $C_w^0(R^+, Y)$ denote the functions from R^+ to Y continuous to the strong and weak topologies of Y, respectively; $C_s^1(R^+, Y)$

and $C^1_w(R^+, Y)$ denote the continuously once-differentiable functions from R^+ to the strong and weak topologies of Y, respectively; $L^\infty_{loc}(R^+, Y)$ is the family of strongly measurable functions from R^+ to Y whose norm is bounded on each finite interval; $\frac{dv}{dt}$ denotes the distribution derivative. Then the assumptions of the above definition may be rewritten:

(1) $u \in C^0_w(R^+, X)$; (2) $u \in C^0_s(R^+, H)$, and $\frac{du}{dt} \in L^\infty_{loc}(R^+, H)$.

(3) $\frac{du}{dt} = f(t, u(t))$, on R^+.

Theorem 2.1: *Let X be a reflexive separable Banach space with $X \subset H \subset X^*$ for a given Hilbert space H. Let T be a hemicontinuous monotone mapping of X into X^* which carries bounded sets of X into bounded sets in X^*. Suppose that $(Tu, u) \to +\infty$ as $||u||_X \to \infty$.*

Then for each u_0 in X such that $T(u_0)$ lies in H, there exists one and only sharp solution of the differential equation

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

on R^+ such that $u(0) = u_0$.

We omit the detailed statement of the corresponding time-dependent result, and pass directly to the generalized method of steepest descent:

Theorem 2.2: *Let X be a reflexive separable Banach space with $X \subset H \subset X^*$ for a given Hilbert space H. Let T be a hemicontinuous monotone mapping of X into X^* such that T maps bounded subsets of X into bounded subsets of X^* while $(Tu, u) \to +\infty$ as $||u||_X \to +\infty$.*

Let c be a C^1 non-increasing function from R^+ to R^+ such that $c(t) \to 0$ as $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$. Let v_0 be an arbitrary element of H, u_0 any element of X such that $T(u_0)$ lies in H.

Then:

(a) The differential equation

$$\frac{du}{dt} = -T(u) - c(t)\,\{u - v_0\}, \qquad t \geq 0,$$

has one and only one sharp solution u on R^+ with $u(0) = u_0$.

(b) As $t \to +\infty$, $u(t)$ converges weakly in X to a solution w_0 of the equation $Tw = 0$. Moreover, $u(t)$ converges strongly in H to w_0. The limit element w_0 is uniquely characterized as the solution of the equation $Tw = 0$ closest to v_0 in H.

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The existence theorems, Theorem 2.1 and its time-dependent generalization which we have not stated, apply directly to the treatment of initial boundary value problems of parabolic type ([3]) and (especially for the time-independent case) give a significant strengthening of the parabolic existence theorems under hypotheses on T which are essentially weaker than those considered in the treatment of *variational* rather than *sharp* solutions. The previous hypotheses (though they can be put in a much more general-looking and untransparent form) have the same force essentially as the following simple assumption:

There exists an exponent p with $1 < p < +\infty$ such that for suitable positive constants c and c_0.

$$\|Tu\|_{X^*} \leq c\{\|u\|_X^{p-1} + 1\};$$
$$(Tu, u) \geq c_0\|u\|_X^p - c.$$

In Theorem (2.1), however, we need only assume that T maps bounded sets of X into bounded sets in X^* and that $(Tu, u) \to +\infty$ as $\|u\|_X \to +\infty$.

Similar considerations apply to the existence theorem which we derive for the abstract wave equation of the form

$$u_{tt} = -Au - T(u_t) - S(u)$$

where T and S are mappings of X into X^*, and u_t, u_{tt} denote the first and second derivatives of u with respect to t. We introduce a class of *sharp* solutions as follows:

Definition: *Let X be a Banach space, H a Hilbert space with $X \subset H \subset X^*$. Let A be a non-negative closed self-adjoint operator in H, $A^{\frac{1}{2}}$ its non-negative square root. Let T and S be mappings of X into X^*.*

Then a function u from R^+ to X is said to be a sharp solution on R^+ of the differential equation

$$u_{tt} = -Au - T(u_t) - S(u)$$

if u satisfies the following five conditions:

(1) *u lies in $C_w^1(R^+, X)$ and in $C_s^1(R^+, H)$.*

(2) *u_{tt} lies in $L_{\text{loc}}^\infty(R^+, H)$.*

(3) *For each t in R^+, $u(t)$ and $u_t(t)$ lie in the domain of $A^{\frac{1}{2}}$, and $A^{\frac{1}{2}}u$ lies in $C_w^1(R^+, H)$, $A^{\frac{1}{2}}u_t$ lies in $C_w^0(R^+, H)$.*

(4) *For each t in R^+, $Au(t)$ lies in X^*, i.e. there exists $y(t)$ in X^* which we denote by $Au(t)$ such that for all w in $D(A^{\frac{1}{2}}) \cap X$, we have*

$$(A^{\frac{1}{2}}u(t), A^{\frac{1}{2}}w) = (y(t), w).$$

Furthermore Au lies in $C_w^0(R^+, X^*)$.

(5) *For almost all t in R^+,*

$$u_{tt}(t) = -Au(t) - T(u_t(t)) - S(u(t)).$$

Theorem 2.3: *Let X be a reflexive separable Banach space, H a Hilbert space with $X \subset H \subset X^*$. Let A be a non-negative closed self-adjoint linear operator in H such that $D(A^{\frac{1}{2}}) \cap X$ is dense both in X and $D(A^{\frac{1}{2}})$, where the latter is given the graph norm. Let T be a hemicontinuous mapping of X into X^* which maps bounded sets into bounded sets, S a Lipschitz mapping of H into H, (where both T and S may be nonlinear). Suppose that $(Tu, u) \to +\infty$ as $||u||_X \to +\infty$ and that T is monotone.*

Then for each u_0 in $D(A^{\frac{1}{2}}) \cap X$ and for each u_1 in $D(A^{\frac{1}{2}}) \cap X$ such that $T(u_1)$ lies in H, there exists one and only one sharp solution u on R^+ of the differential equation

$$u_{tt} = -Au - T(u_t) - S(u)$$

which satisfies the initial conditions

$$u(0) = u_0, \qquad u_t(0) = u_1.$$

Abstract wave equations of the above form with S linear but with time-dependent terms were studied by LIONS and STRAUSS [24] who obtained *variational* solutions for similar initial value problems but under growth conditions for T like those disccused above in connection with the first order case.

We now turn from operators T mapping X into X^* to the consideration of J-monotone operators T from X to X. These are defined as follows:

Definition: *Let X be a Banach space, q a continuous strictly increasing function from R^+ to R^+ such that $q(0) = 0$ and $q(r) \to +\infty$ as $r \to +\infty$. Then a mapping J of X into X^* is said to be a duality mapping with gauge function q if the following conditions hold for all u in X:*

$$(Ju, u) = ||u|| \,.\, ||Ju||; \qquad ||Ju|| = q(||u||).$$

Definition: *Let X be a Banach space, J a duality mapping of X into X^*. If T is a mapping with domain $D(T)$ in X and with range in X, then T is said to be J-monotone if for all u and v in $D(T)$,*

$$(T(u) - T(v), J(u - v)) \geq 0.$$

The definition of J-monotone mapping was first given and applied in BROWDER [10] and results on J-monotone mappings have been established in BROWDER [15] and BROWDER—FIGUEIREDO [19]. The concept of J-monotonicity is intimately linked to that of non-expansiveness of a mapping from X to X, where U is said to be *non-expansive* if for all u and v of X,

$$||U(u) - U(v)|| \leq ||u - v||.$$

For every non-expansive mapping U, $T = I - U$ is J-monotone for any duality mapping J of X into X^*. On the other hand, if the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -T(u), \qquad t \geq 0,$$

has a solution u on R^+ with $u(0) = u_0$ for every u_0 in $D(T)$ and if we set $U(t)\,u_0 = u(t)$, then the non-expansiveness of all the operators $U(t)$ is equivalent to the J-monotonicity of T. In particular, if L is a closed densely defined linear operator in X, then L is the generator of a C_0 semigroup of nonexpansive linear operators $U(t)$, (i.e. $||U(t)|| \leq 1$, $t > 0$) if and only if L satisfies both of the following conditions:

(1) $(-L)$ is J-monotone for any duality mapping J of X into X^*.

(2) $(-L + I)$ has all of X as its range.

We present results on J-monotone operators T of two types. First, with mild regularity assumptions on T and very weak assumptions on the space X. Second, with weak assumptions on the operator T (comparable to those in the Hilbert space case) but with fairly drastic restrictions on the Banach space X.

Definition: *A mapping T of X into X is said to be weakly once-differentiable at u_0 in X if there exists a bounded linear operator B such that for all x in X and all y in X^*,*

$$(T(u_0 + hx), y) = (T(u_0), y) + h(Bx, y) + R_{x,y}(h)$$

where for each fixed y in X^*,

$$h^{-1}R_{x,y}(h) \to 0, \qquad \text{as} \qquad h \to 0,$$

uniformly in x on the unit ball of X.

Theorem 2.4: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let T be a nonlinear mapping of X into X which is weakly once differentiable and locally Lipschitzian at each point of X. Suppose that there exists a constant c in R^1 such that for all u and v in X:*

$$(T(u) - T(v), J(u - v)) \leq c||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 in X, there exists one and only one strongly C^1 function u from R^+ to X which satisfies the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = T(u), \qquad t \geq 0,$$

with $u(0) = u_0$.

Theorem 2.5: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let L be a closed densely defined linear operator in L which is the infinitesimal generator of a C_0 semigroup of nonexpansive linear operators in X. Let T_0 be a nonlinear mapping of X into X which is weakly once-differentiable and locally Lipschitzian in a neighborhood of each point of $D(L)$, and such* *that T_0 maps bounded subsets of X into bounded subsets of X. Suppose also that there exists a constant c in R^1 such that for all u and v in X,*

$$(T_0(u) - T_0(v), J(u - v)) \leq c||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 in $D(L)$, there exists one and only one strongly C^1 function u from R^+ to X with $u(t)$ in $D(L)$ for all tvin R^+ such that u satisfies the differential equation

$$\frac{du}{dt} = Lu + T_0(u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$.

Theorem 2.6: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X. Let T_0 be a mapping of $R^+ \times X$ into X which maps bounded subsets of $R^+ \times X$ into bounded subsets of X. Suppose that for each fixed t in R^+, $T_0(t, u)$ is weakly once-differentiable and locally Lipschitzian on a neighborhood of each point of $D(L)$. Suppose further that both of the following conditions are satisfied:*

(a) There exists a continuous function c from R^+ to R^1 such that for all u and v in $D(L)$ and all t in R^+,

$$(T_0(t, u) - T_0(t, v), J(u - v)) \leq c(t)\, ||u - v|| \,.\, ||J(u - v)||.$$

(b) For each fixed u in $D(L)$, $T_0(t, u)$ is weakly once differentiable in t on R^+. There exist two continuous functions k: $R^+ \to R^+$ and q: $R^+ \times R^+ \to R^+$ such that for all u in $D(L)$ and all t in R^+,

$$\left\| \left(\frac{\partial}{\partial t} T_0\right)(t, u) \right\| \leq k(t)\, ||Lu|| + q(t, ||u||).$$

Then for each u_0 in $D(L)$, there exists one and only one strongly C^1 function u from R^+ to X with $u(t)$ lying in $D(L)$ for all t in R^+ such that u is a solution of the differential equation

$$\frac{du}{dt} = Lu + T_0(t, u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$ holds.

For this case, we obtain the following variant of the method of steepest descent:

Theorem 2.7: *Let X be a Banach space with a continuous duality mapping J of X into X^*. Let T be a mapping with domain and range in X which lies in one of the following two classes:*

(1) *T is a J-monotone mapping of X into X which is weakly once-differentiable and locally Lipschitzian at each point of X.*

(2) $T = -L + T_0$, *where* L *is a closed linear operator in* X *which is the infinitesimal generator of a* C_0 *semigroup of nonexpansive operators in* X, T_0 *is a nonlinear J-monotone mapping of* X *into* X *which carries bounded sets into bounded sets and such that* T_0 *is weakly once-differentiable and locally Lipschitzian on a neighborhood of each point of* $D(L)$.

Suppose that there exists $R > 0$ *such that for* u *in* $D(T)$ *with* $||u|| = R$, $(Tu, Ju) \geq 0$.

Let c *be a nonincreasing* C^1 *function from* R^+ *to* R^+ *with* $c(t) \to 0$ *as* $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$. *Let* u_0 *be any element of* $D(T)$ *with* $||u_0|| \leq R$, *and let* v_0 *be any element of* X *with* $||v_0|| < R$.

Then:

(*a*) *The differential equation*

$$\frac{du}{dt} = -T(u) - c(t)\{u - v_0\}$$

has one and only one solution u *on* R^+ *with* $u(0) = u_0$.

(*b*) *For each such solution* u *on* R^+, *we have*

$$||T(u(t))|| \to 0$$

as $t \to +\infty$.

(*c*) *Suppose that in addition to the preceding hypotheses,* T *satisfies the following condition:*

(*C*) *For each* $M > 0$, *there exists a compact mapping* C *of* X *into* X *and a continuous strictly increasing function* p *from* R^+ *to* R^+ *with* $p(0) = 0$ *such that for all* u *and* v *of* $D(T)$ *with* $||u|| \leq M$, $||v|| \leq M$,

$$||T(u) - T(v)|| \geq p(||u - v||) - ||C(u) - C(v)||.$$

Then $u(t)$ *converges strongly in* X *as* $t \to +\infty$ *to a solution* v_0 *of the equation* $Tv_0 = 0$.

As a consequence of Theorem 2.7, we have the following existence theorem for solution of the equation $Tv = w$.

Theorem 2.8: *Let* X *be a Banach space with a continuous duality mapping* J *of* X *into* X^*, *and let* T *be a J-monotone mapping which is in one the two classes* (1) *or* (2) *of Theorem* (2.7). *Then:*

(1) *Let* B_R *be the closed ball of radius* $R > 0$ *about the origin in* X. S_R *its boundary. If for some* $R > 0$, $(Tu, Ju) \geq 0$ *for all* u *in* $D(T) \cap S_R$, *then* 0 *lies in the strong closure of* $T(B_R \cap D(T))$. *In particular, if* $T(B_R \cap D(T))$ *is closed in* X, *then the equation* $Tv = 0$ *has a solution* v_0 *with* $||v_0|| \leq R$.

(2) *Suppose that T is J-coercive, i.e.*

$$(Tu, Ju)/||Ju|| \to +\infty, \qquad (||u|| \to +\infty).$$

Then the range of T is dense in X.

(3) *If T is J-coercive and satisfies condition (C) of part (c) of Theorem 2.7, then the range of T is the whole of X.*

(4) *If X is reflexive and T is J-coercive as well as demiclosed (i.e. for any weakly convergent sequence $u_j \to u$ with Tu_j converging strongly to w, u lies in $D(T)$ and $Tu = w$), then the range of T is all of X.*

(5) *If X is strictly convex and T is J-coercive, the set*

$$K_w = \{v | v \in D(T), Tv = w\}$$

, for a fixed w in X, is a closed convex subset of X.

We now restrict the class of Banach spaces X, and thereby can eliminate the regularity conditions imposed upon T in the preceding results. Our basic hypothesis upon X is the following:

Definition: *X is said to satisfy the conditions (P) if the following two conditions hold:*

(1) *There exists a duality mapping J of X into X^* which is continuous and also weakly continuous (i.e. continuous in the weak topology of X and X^*).*

(2) *There exists an increasing sequence $\{F_j\}$ of finite dimensional subspaces of X whose union is dense in X, and a corresponding sequence $\{P_j\}$ of projections of X such that the range of each P_j is the corresponding F_j and for each j, $||P_j|| = 1$.*

The properties (P) were applied in BROWDER—FIGUEIREDO [19] to obtain an existence theorem for nonlinear functional equations involving J-monotone operators. Aside from Hilbert spaces, the most important class of concretely defined Banach spaces which satisfy the conditions (P) are the sequence spaces l^p for $1 < p < +\infty$, as was shown in [10]. The restrictive condition in the pair of conditions (P) is the first which does not hold for any L^p space with $p \neq 2$ on the line. Property (2) seems to hold for all examples of separable Banach spaces familiar to the writer.

Theorem 2.9: *Let X be a reflexive Banach space which is strictly convex and satisfies the conditions (P). Let J be any duality mapping of X into X^*. Let T be a mapping of X into X which is hemicontinuous and locally bounded, and for which there exists a constant c in R^1 such that for all u and v of X*

$$(T(u) - T(v), J(u - v)) \leq c||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 in X, there exists one and only weakly C^1 function u from R^+ to X which satisfies the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = T(u), \qquad t \geq 0,$$

and the initial condition $u(0) = u_0$.

Theorem 2.10: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), J a duality mapping of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X. Let T_0 be a mapping of X into X which is hemicontinuous, maps bounded subsets of X into bounded subsets of X, and for which there exists a constant c in R^1 such that for all u and v of X,*

$$(T_0(u) - T_0(v), J(u - v)) \leq c||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 in $D(L)$, there exists one and only one sharp solution u on R^+ of the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = Lu + T_0(u), \qquad t \geq 0,$$

with $u(0) = u_0$.

By a *sharp solution*, we mean a function u from R^+ to X which lies in $C^1_w(R^+, X)$ with $u(t)$ in $D(L)$ for all t in R^+ and with Lu in $C^0_w(R^+, X)$.

A time dependent generalization of Theorem 2.10 is the following:

Theorem 2.11: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P) and let J be a duality mapping of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive linear operators in X. Let T_0 be a mapping of $R^+ \times X$ into X which carries bounded sets into bounded sets and satisfies the following three conditions:*

(1) *For each fixed t in R^+, $T_0(t, \cdot)$ is a hemicontinuous mapping of X into X. For each fixed u in X, $T_0(\cdot, u)$ is a continuous mapping from R^+ to the weak topology of X.*

(2) *There exists a continuous function c from R^+ to R^1 such that for all u and v in X and all t in R^+:*

$$(T_0(t, u) - T_0(t, v), J(u - v)) \leq c(t)\, ||u - v|| \,.\, ||J(u - v)||.$$

(3) *For each fixed u in $D(L)$, $T_0(t, u)$ is weakly once differentiable in t from R^+ to X, and its derivative satisfies the inequality*

$$\left\| \left(\frac{\partial}{\partial t} T_0\right)(t, u) \right\| \leq q(t, ||u||)$$

for all u in $D(L)$ and a continuous function q from $R^+ \times R^+$ to R^+.

Then for each u_0 in $D(L)$, there exists one and only one sharp solution u on R^+ of the differential equation

$$\frac{du}{dt} = Lu + T_0(t, u), \qquad t \geq 0,$$

with $u(0) = u_0$.

The variant of the generalized method of steepest descent which holds for this case is the following:

Theorem 2.12: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), and let J be any duality mapping of X into X^*. Let L be a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X. Let T_0 be a mapping of X into X which is hemicontinuous and locally bounded. If L is unbounded, we suppose in addition that T_0 maps bounded sets of X into bounded sets of X.*

Suppose that T_0 is J-monotone, and that there exists $R > 0$, such that $(Tu, Ju) \geq 0$ for all u in $D(T)$ with $||u|| = R$.

Let c be a continuous nonincreasing C^1 function from R^+ to R^+ such that $c(t) \to 0$ as $t \to +\infty$, $\int_0^\infty c(s)\, ds = +\infty$. Let u_0 be any element of $D(L)$ with $||u_0|| \leq R$, and let v_0 be any element of X with $||v_0|| < R$.

Then:

(a) There exists exactly one sharp solution u on R^+ of the differential equation

$$\frac{du}{dt} = Lu - T_0(u) - c(t)\, \{u - v_0\}, \qquad t \geq 0,$$

with $u(0) = u_0$.

(b) For each such solution,

$$||-Lu(t) + T_0(u(t))|| \to 0$$

as $t \to +\infty$.

(c) For each such solution, $u(t)$ converges strongly in X to a solution v_0 of the equation $Tv_0 = 0$, as $t \to +\infty$.

A consequence of Theorem (2.12) is the following existence theorem for solutions of nonlinear functional equations involving J-monotone operators.

Theorem 2.13: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), J a duality mapping of X into X^*. Let T be a mapping with domain and range in X which lies in one of the following two classes:*

(a) T is a hemicontinuous locally bounded J-monotone mapping of X into X.

(b) $T = -L + T_0$, where L is a closed linear operator in X which is the infinitesimal generator of a C_0 semigroup of nonexpansive operators in X, and

T_0 *is a hemicontinuous J-monotone mapping of X into X which carries bounded sets into bounded sets.*

Then:

(1) *If for a given* $R > 0$ *and for all u in* $D(T)$ *with* $||u|| = R$, $(Tu, Ju) \geq 0$, *then the set*

$$K = \{v | v \in D(L), Tv = 0, ||v|| \leq R\}$$

is a nonempty closed convex subset of X.

(2) *If T is J-coercive, then the range of T is all of X.*

The existence of a solution u_0 of the equation $Tu_0 = 0$ in case (*a*) was previously established in Browder—Figueiredo [19].

Let us turn finally to nonlinear equations of evolution involving J-monotone operators without a differentiability assumption on the dependence of f on t.

First, we have the following theorem which extends the similar result in Hilbert space proved in Browder [4]:

Theorem 2.14: *Let X be a reflexive strictly convex Banach space which satisfies the conditions (P), and let J be a duality mapping of X into* X^*. *Suppose that f is a mapping of* $R^+ \times X$ *into X which carries bounded subsets of* $R^+ \times X$ *into bounded sets in X. Suppose that f satisfies the following two conditions:*

(1) *For each fixed t in* R^+, $f(t, \cdot)$ *is a hemicontinuous mapping of X into X. For each fixed u in X,* $f(\cdot, u)$ *is a continuous mapping of* R^+ *into the weak topology of X.*

(2) *There exists a continuous function c from* R^+ *to* R^1 *such that for all t in* R^+ *and all u and v of X,*

$$(f(t, u) - f(t, v), J(u - v)) \leq c(t)\, ||u - v|| \,.\, ||J(u - v)||.$$

Then for each u_0 *in X, there exists one and only one solution u in* $C^1_w(R^+, X)$ *of the differential equation*

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

which satisfies the initial condition $u(0) = u_0$.

A corresponding extension of the existence theorems for mild solutions of nonlinear equations of evolution in Hilbert space involving unbounded linear operators, as proved in Browder [4] and Kato [21], is based upon the following natural extension of the definition of *mild solution*:

Definition: *Let X be a Banach space,* $\{L(t) \mid t \in R^+\}$ *a family of closed linear operators in X, f a mapping of* $R^+ \times X$ *into X. Suppose that the time-dependent linear problem*

$$\frac{du}{dt}(t) = L(t)\,u(t), \qquad t \geq s,$$

$u(s) = u_0$

has one and only one strongly continuous solution $u(t) = U(t, s)\,u_0$ *for each* $s \geq 0$ *and each* u_0 *in* $D(L(s))$, *where* $U(t, s)$ *is a bounded linear operator in* X *for each* s *and* t *in* R^+ *with* $s \leq t$.

Then a function u *from* R^+ *to* X *is said to be a mild solution of the nonlinear differential equation*

$$\frac{du}{dt} = L(t)\,u + f(t, u), \qquad t \geq 0,$$

if u *is a strongly continuous function from* R^+ *to* X *which is a solution of the nonlinear integral equation:*

$$u(t) = U(t, 0)\,u_0 + \int_0^t U(t, s)\,f(s, u(s))\,ds, \qquad t \geq 0.$$

Theorem 2.15: *Let* X *be a reflexive strictly convex Banach space which satisfies the conditions* (P), J *a duality mapping of* X *into* X^*. *Let* $\{L(t) \mid t \in R^+\}$ *be a family of closed linear operators in* X, *with each* $L(t)$ *the infinitesimal generator of a* C_0 *semigroup of nonexpansive operators in* X. *Suppose also that for each* $s \geq 0$ *and each* u_0 *in* $D(L(s))$, *the time-dependent linear problem*

$$\frac{du}{dt}(t) = L(t)\,u(t), \qquad t \geq s,$$

$u(s) = u_0$

has one and only one solution u *in* $C^1_w((s, \infty); X)$.

Let f *be a mapping of* $R^+ \times X$ *into* X *which maps bounded subsets of* $R^+ \times X$ *into bounded subsets of* X *and satisfies the two conditions* (1) *and* (2) *of Theorem* (2.14).

Then there exists for each u_0 *in* X, *one and only one mild solution* u *on* R^+ *of the nonlinear equation of evolution*

$$\frac{du}{dt} = L(t)\,u + f(t, u), \qquad t \geq 0,$$

with $u(0) = u_0$.

Section 3: We now turn to the problem of the existence of periodic solutions of equations of the form

$$\frac{du}{dt} = f(t, u) \tag{3.1}$$

where $f(t, u)$ is periodic in t of period p, i.e. $f(t + p, u) = f(t, u)$ for all t in R^+. We seek to find periodic solutions of period p. We shall present here some of the simpler results given in BROWDER [15].

Definition: *A function V from the Banach space X to R^+ is said to be a Lyapounov function for the equation* (**3.1**), *where f is a mapping of $R^+ \times X$ into X, if the following conditions are valid:*

(1) *V is a convex function on X, with $V(0) = 0$, $V(u) > 0$ for $u \neq 0$, and the level sets of V are bounded and uniformly convex, i.e. given $R > 0$, $d > 0$ there exists $R_1 < R$ such that if $V(u_0) \leq R$, $V(u_1) \leq R$, with $||u_0 - u_1|| \geq d$, then:*

$$V((u_0 + u_1)/2) \leq R_1.$$

(2) *There exists a continuous mapping S of X into X^* which is a subderivative of V, i.e. for all u and v in X*

$$V(u) - V(v) \geq (S(v), u - v).$$

(3) *For each pair u and v in X and all t in R^+,*

$$(f(t, u) - f(t, v)\ \mathrm{S}\ (u - v)) \leq 0.$$

(4) *There exists $R_0 > 0$ such that for all t in R^+ and all u in X with $||u|| \geq R_0$,*

$$(f(t, u), S(u)) \leq 0.$$

Theorem 3.1: *Let X be a reflexive Banach space, f a mapping of $R^+ \times X$ into X such that for all u_0 in X, the differential equation*

$$\frac{du}{dt} = f(t, u), \qquad t \geq 0,$$

has exactly one solution with $u(0) = u_0$.

Suppose that $f(t, u)$ is periodic in t of period $p > 0$, and suppose that there exists a Lyapounov function for this equation in the sense of the above definition.

Then the equation (**3.1**) *has a periodic solution of period p.*

As an application of this result, we have the following:

Theorem 3.2: *Let X be a uniformly convex Banach space, J a duality mapping of X into X^*. Let f be a mapping of $R^+ \times X$ into X such that the equation*

$$\frac{du}{dt} = f(t, u)$$

has one and only one solution on R^+ with $u(0) = u_0$, for any given u_0 in X. Suppose further that for each t in R^+,

$$(f(t, u) - f(t, v), J(u - v)) \leq 0,$$

and that there exists $R > 0$ such that for all t in R^+ and all u in X with $||u|| \geq R$,

$$(f(t, u), Ju) \leq 0.$$

Then if $f(t, u)$ is periodic in t of period $p > 0$, there exists a solution of the differential equation which is periodic of period p.

Extensions are given in [ ] to more general nonlinear equations of evolution of the types considered in Sections 1 and 2 above. The proofs are all based upon the following simple fixed point theorem:

Theorem 3.3: *Let X be a reflexive Banach space, V a convex continuous function from X to R^+ such that $V(0) = 0$, $V(u) > 0$ for $u \neq 0$. Suppose that the level sets of V are bounded and uniformly convex. Let U be a mapping of a closed convex subset C of X into C such that for all u and v of C,*

$$V(U(u) - U(v)) \leq V(u - v).$$

Then U has a fixed point in C.

The proof of Theorem 3.3 uses an argument of Brodski and Milman [1], which was applied in the case in which V is a function of the norm in an uniformly convex space by Browder [9] and Kirk [22]. Similar fixed point theorems with weakened hypotheses can be established in Hilbert spaces and Banach spaces having weakly continuous duality mappings J by using the fact that for every nonexpansive mapping U, $T = I - U$ is J-monotone, (cf [10], [17]).

Theorem 3.2 is an extension of a result in Hilbert space given by the writer in [8].

We remark in conclusion that the most general form of aplication of Theorem (3.3) to initial value problems can be put in the following abstract form:

Theorem 3.4: *Let X be a reflexive Banach space, C a closed convex bounded subset of X. Let $\{U(t, s) \mid t \geq s\}$ be a family of transition operators on C, i.e. for $r \leq s \leq t$, $U(t, r) = U(t, s)\, U(s, r)$, where each $U(t, s)$ is a (possibly) nonlinear nonexpansive mapping of C into itself. Suppose further that there exists a convex function V from X to R^+ such that $V(0) = 0$, $V(u) > 0$ for $u \neq 0$, and with the level surfaces of V uniformly convex, such that for all t and s, $(s \leq t)$ and all u and v in C,*

$$V(U(t, s)\, u - U(t, s)\, v) \leq V(u - v).$$

Suppose that the transition operators $U(t, s)$ are periodic of period $p > 0$, in the sense that for every $s < t$ in R^+, $U(t + p, s + p) = U(t, s)$.

Then there exists u_0 in C such that for every t in R^+, $U(t, 0)\, u_0$ is periodic in t of period p.

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