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INVESTIGATION OF THE SOLUTIONS OF DIFFERENTIAL EQUATIONS ON AN INFINITE INTERVAL AND THE FIXED POINT THEOREMS

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The fundamental question which is to solve in the theory of the differential equations is the question of existence. It can be solved by various methods, chosen following the made assumptions and the expected properties of the solution. In the last time the methods based on the theorems of the fixed points seem to very efficient. Those theorems serve as a very important and convenient way, and we can state that they are the most elegant, for the proof of the existence of the solution determined for instance by the initial conditions (not only for the proof of the local existence, but also of the existence in large). Then there are boundary-value problems, the linear problems, the problems of the existence of the periodic or almost periodic solutions, the existence of bounded solutions, of monotone solutions or of the solutions having other required properties. In all this problems the theorems of the fixed point have been used with a great succes. I could quote a very long list of the works concerning with those problems (to begin with the works of G. D. Birkhoff, O. D. Kellog, R. Cacciopoli, Schauder, Leray to the last works of Krasnoselski, Browder, Cesari, Hale, Urabe, Knobloch, Conti, Kakutani, Lasota, Opial, Haimovici, Bielecki, Corduneanu and others).

I will consider the theorem of Schauder and indicate some of the variants which are very convenient especially in the case when the existence of solutions is to be proved with the required properties in an infinite interval.

We find the theorem of Schauder quoted in the literature essentially in two forms of what the following form seems to be more convenient for application:

Let M be a convex and closed set of a Banach space. Let T be a continuous operator on M such that $TM \subset M$ and TM is (relatively) compact. Then T has at least one fixed point in M.

In utilizing this theorem one takes for M generally the closed sphere which is evidently a convex set. Then there are three things to prove: The continuity of T on M, the transformation of M by T in itself and the compactness of TM. It is chiefly the compactness which gives many difficulties. It can be proved often by use of the theorem of Arzela, but this theorem requires that the domain of the definition of the functions of TM be bounded. If this domain is not bounded, it is possible to use the theorem of HAUSDORF of the existence of the ε-net.

In the following lines I will consider the cases where the interval of the definition of the functions of TM is not bounded and I will show, in using the notion of the quasiconvergence, how to evade the difficulties which can arise. In the first place I shall try to explain the ideas on a concrete Banach space and to prepare everything in order to their application in the dofferential equations of the n-th order.

Let A_{n-1} be the set of all functions which have, on the interval J, the continuous derivatives till the order $n - 1$ inclusively.

I must give some definitions.

D$_1$. *Let $f_k(x)$, $k = 1, 2, \ldots,$ be the functions of A_{n-1}. We will say that the sequence $\{f_k(x)\}$ converges quasi-uniformly (or shortly q-converges) to the function $f(x)$ on J, if for every $x \in J$ and $i = 0, 1, \ldots, n - 1$, $\lim_{k \to \infty} f_k^{(i)}(x) = f^{(i)}(x)$. We write $f_k \xrightarrow{q} f$.*

It is evident that every subsequence of a sequence which q-converges to $f(x)$, q-converges to $f(x)$.

D$_2$. *Let S_{n-1} be the Banach space of all functions of A_{n-1}, which have the bounded derivatives till the order $n - 1$ inclusively. The norm is given by the formula*

$$||f(x)|| = \max_{0 \leq i \leq n-1} \{\sup_J |f^{(i)}(x)|\}.$$

It is easily to shown that the convergence in this norm implicates the q-convergence, and that is essential for us.

D$_3$. *The infinite set $M \subset S_{n-1}$ is said to be q-compact in S_{n-1} if every sequence extracted from M contains a subsequence q-convergent to a function of S_{n-1}.*

It is to be noted that the limit of a q-convergence sequence of S_{n-1} ought no to be of S_{n-1}.

D$_4$. *We will say that the set $M \subset S_{n-1}$ is q-closed if the following implication holds: $\{f_k \in M, f_k \xrightarrow{q} f\} \Rightarrow \{f \in M\}$.*

D$_5$. *We will say that the functions of the set $M \subset S_{n-1}$ are uniformly bounded on J by a number K, if $|f^{(i)}(x)| \leq K$ for every $x \in J$, $i = 0, 1, \ldots, n - 1$ and for every $f(x) \in M$. We will say that the functions of M are equicontinuous on J if holds: for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such, that for every $f(x) \in M$, for $i = 0, 1, \ldots, n - 1$ and for $|x - x'| < \delta(\varepsilon)$ holds: $|f^{(i)}(x) - f^{(i)}(x')| < \varepsilon$.*

It is easy to prove [2]:

Lemma 1. *If the functions of an infinite set* $M \subset S_{n-1}$ *are uniformly bounded and equicontinuous on* J, *then* M *is* q*-compact in* S_{n-1}.

With the help of the q-convergence we can define the q-continuity of an operator T on S_{n-1} (or on a set $M \subset S_{n-1}$).

D_6. *An operator* T *on* S_{n-1} *into* S_{n-1} *(on* M *into* S_{n-1}*) is* q*-continuous on* S_{n-1} *(on* M*) iff the following implication holds*: $\{f_k \xrightarrow{q} f,\ f_k,\ f \in S_{n-1}\} \Rightarrow \Rightarrow \{\|Tf_k - Tf\| \to 0 \text{ for } k \to \infty\}$, $(\{f_k \xrightarrow{q} f,\ f_k,\ f \in M\} \Rightarrow \{\|Tf_k - Tf\| \to 0 \text{ for } k \to \infty\})$.

The q-continuous operator has the following (for us very important) property):

The q-continuous operator is also continuous.

Lemma 2. *If* $M \subset S_{n-1}$ *is* q*-compact in* S_{n-1} *and if* T *is* q*-continuous operator on* M *into* S_{n-1}, *then* $TM \subset S_{n-1}$ *is compact in* S_{n-1}. (See [2]).

From this property follows immediately the first variant of the theorem of SCHAUDER [2].

Theorem 1. *Let* T *be an operator* q*-continuous on* $M \subset S_{n-1}$, *let* M *be convex, closed and* q*-compact, and let* $TM \subset M$. *Then* T *has at least one fixed point on* M.

The assumption of the theorem of SCHAUDER mentioned above that TM is compact, is substitued here by the assumption that T is q-continuous on M and that M is q-compact. For which follows the following lemmas are very important.

Lemma 3. *Let the functions of* $M \subset S_{n-1}$ *be uniformly bounded and equicontinuous on* J. *Then the functions of the convexe hull* $\hat{M}$ *of* M *and also the functions of the closure* $\overline{M}$ *of* M *are uniformly bounded and equicontinuous on* J [2].

Lemma 4. *If* $M \subset S_{n-1}$ *is convex, then the closure* $\overline{M}$ *of* M *is also convexe.*

Now we are able to prove the

Theorem 2. *Let* T *be an operator* q*-continuous on* S_{n-1}. *Let* $M \subset S_{n-1}$ *be a convex and closed set. Let* $TM \subset M$ *and let the functions of* TM *be uniformly bounded and equicontinuous on* J. *Then* T *has at least one fixed point on* M [2].

The proof of this theorem is based on the fact that, following the lemmas 3 and 4, the closure of the convex hull $\overline{\widehat{TM}} = N$ of TM is a set of the functions which are uniformly bounded and equicontinuous on J. Following the lemma 1 N is q-compact. But this set is convexe and closed and $N \subset M$. From here we have: $TN \subset TM \subset \overline{\widehat{TM}} = N$. The application of the theorem 1 finishes the proof.

For the applications the following theorems are more convenient [2].

Theorem 3. *Let T be an operator on S_{n-1} such that the following implication takes place:*

$$\{f_k, f \in S_{n-1}, f_k \xrightarrow{q} f, \{||f_k||\} \text{ bounded}\} \Rightarrow \{||Tf_k - Tf|| \to 0 \text{ for } k \to \infty\}.$$

Futher, let $M \subset S_{n-1}$ be a convex and bounded set and $TM \subset M$. Let the functions of TM be uniformly bounded and equicontinuous on J. Then T has at least one fixed point in M.

Theorem 4. *Let $M \subset S_{n-1}$ be a convex and bounded set, let T be an operator q-continuous on M such that $TM \subset M$. Let the functions of TM be uniformly bounded and equicontinuous on J. Then T has at least one fixed point in M.*

The proof of those two theorems is not different from the proof of the theorem 2.

We shall now proceed to the application of those theorems on the differential equations.

First of all we need the following lemma.

Lemma 5. *([1] and [2]). Let $Q(x)$ be a function, which is, on the interval (a, ∞), $-\infty \leqq a$, continuous and non-negative in such a way that it is not identically zero on none of the subintervals of the interval (a, ∞). Then the differential equation*

$$\text{(A)} \qquad u^{(n)} + (-1)^{n+1}Q(x)\, u = 0$$

has a solution $u(x)$ having the following properties:

$$(\mathrm{V}_1) \quad \begin{cases} (-1)^k u^{(k)}(x) > 0 \quad \text{or} \quad (-1)^{k+1} u^{(k)}(x) > 0, \qquad k = 0,1, \ldots, n-1, \\ \lim\limits_{x \to \infty} u^{(k)}(x) = 0, \quad k = 1, 2, \ldots, n-1, \\ \lim\limits_{x \to \infty} u(x) \quad \text{exists and is finite.} \end{cases}$$

It holds yet that $\lim\limits_{x \to \infty} u(x) = 0$ iff $\int^{\infty} x^{n-1}Q(x)\,\mathrm{d}x = \infty$. If $\int^{\infty} x^{n-1}Q(x)\,\mathrm{d}x < \infty$, there is exactly one solution (excepted the linear dependence) having the properties (V_1). In this case $\lim\limits_{x \to \infty} u(x) \neq 0$, and we will say that $u(x)$ has the properties (V).

Let us now consider the differential equation

$$\text{(B)} \qquad y^{(n)} + (-1)^{n+1}B(x, y, y', \ldots, y^{(n-1)})\, y = 0.$$

Has this equation a solution having the properties (V) when the function B has similar properties as those of $Q(x)$? The following theorem gives an affirmative answer [2].

Theorem 5. *Let be the following conditions fulfilled:*

1. *The function* $B(x, \mathbf{u})$, $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$ *(*$\mathbf{u}$ *being a vector with the terms* $u_0, u_1, \ldots, u_{n-1}$*) is in the domain*

$$\Omega : a < x < \infty, \qquad -\infty < u_i < \infty, \qquad i = 0, 1, \ldots, n-1,$$

continuous in $(x, \mathbf{u})$ *and non-negative such that for every point* $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \neq (0, 0, \ldots, 0)$ *the function* $B(x, \mathbf{c})$ *equals identicaly to zero in none of the subintervals of the interval* (a, ∞).

2. $B(x, \mathbf{u})$ *is monoton in every one of his variables* u_i, $i = 0,1, \ldots, n-1$, *for* $u_i \geqq 0$ *as well as for* $u_i < 0$ *(the monotony for* $u_i \geqq 0$ *can be different from that for* $u_i < 0$*).*

3. *For every point* $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1})$ *is*

$$\int^{\infty} x^{n-1} B(x, \mathbf{c})\, dx < \infty.$$

4. $$\lim_{k \to \infty} \frac{1}{k} \int^{\infty} x^{n-2} B(x, \mathbf{c})\, dx = 0 \text{ for } |\mathbf{c}| = \sum_{i=0}^{n-1} |c_i| \leqq k.$$

Then through every point (x_0, y_0), $x_0 \in (a, \infty)$, $y_0 \neq 0$, *passes at least one solution* $z(x)$ *of the equation* (B) *having the properties* (V) *on the interval of his existence (which is not smaller then* $\langle x_0, \infty)$*).*

We draw a sketch of the proof. Let be $J = \langle x_0, \infty)$, $x_0 > a$. We are looking for the solution $z(x)$ of (B) in the space S_{n-1}. Let be $G_k = \{f(x) \in S_{n-1} | \, ||f(x)|| \leqq k\}$ (the sphere closed). Then from the monotony of $B(x, \mathbf{u})$ in u_i follows that for every $f(x) \in G_k$

$B(x, f(x), f'(x), \ldots, f^{(n-1)}(x)) = B(x, \mathbf{f}(x)) \leqq B(x, \vartheta_0, \vartheta_1, \ldots, \vartheta_{n-1}) = B(x, \boldsymbol{\theta})$ where ϑ_i means one of the numbers 0, k, $-k$ according to the monotony of B in u_i. $B(x, \boldsymbol{\theta})$ is a majorante integrable.

In view of 3. we have

$$\int^{\infty} x^{n-1} B(x, \mathbf{f}(x))\, dx \leqq \int^{\infty} x^{n-1} B(x, \boldsymbol{\theta})\, dx < \infty. \tag{1}$$

Then for the equation

$$y^{(n)} + (-1)^{n+1} B(x, \mathbf{f}(x))\, y = 0 \tag{2}$$

holds the lemme 5. There exists just one solution $u(x)$ of this equation which passes through the point (x_0, y_0) having the properties (V) on J. With the help of this we can define an operator T on G_k in this way: If $f(x) \in G_k$, then $Tf(x) = u(x)$ is the unique solution of (2) having the properties (V) on J and which passes through the point (x_0, y_0). This solution is also a solution of the integral equation

$$u(x) = y_0 - (-1)^{n+1} \int_{x_0}^{\infty} \frac{(x_0 - t)^{n-1}}{(n - 1)!} B(t, f(t))\, u(t)\, dt +$$

$$+ (-1)^{n+1} \int_{x}^{\infty} \frac{(x - t)^{n-1}}{(n - 1)!} B(t, f(t))\, u(t)\, dt.$$

With the help of 3^0 we can prove that $TG_k \subset S_{n-1}$ and with the help of 4^0 we can prove the existence of such a number k_0 that $TG_{k_0} \subset G_{k_0}$. The sphere G_{k_0} is evidently closed and convexe. It is easy to prove that it is also q-closed. From 3^0, (1) and from the Lebesgue's theorem follows the q-continuity of T on G_{k_0}. Then we prove that the functions of TG_{k_0} are uniformly bounded and equicontinuous on J. The application of the theorem 4 gives the existence of a solution of (B) having the properties (V) on J. Next it can be easily proved that this solution can be extended to an interval (b, ∞), $a \leqq b < x_0$ and this extended solution has the properties (V) on (b, ∞).

Let us now return a little to the equation (A). If we suppose that $\int^{\infty} x^{n-1}Q(x)\, dx < \infty$, then there exists just one solution $u(x)$ of (A) having the properties (V) and such that $\lim_{x \to \infty} u(x)\, m_0 \neq 0$, m_0 beeing a real number choosen arbitrary. On the basis of this we can prove the

Theorem 6. *Let the conditions* 1^0, 2^0 *and* 3^0 *of the theorem* 5 *be fulfilled. Let* m_0 *be an arbitrary real number different from zero. Then there exists at least one solution* $z(x)$ *of* (B) *which has, on the interval of his existence, the propertiets* (V) *and for which* $\lim_{x \to \infty} z(x) = m_0$.

The proof is similar to that of the theorem 5. We define the operator T on the sphere G_k : if $f(x) \in G_k$, then $Tf(x) = u(x)$, where $u(x)$ is the solution of the equation

$$y^{(n)} + (-1)^{n+1} B(x, f(x))\, y = 0$$

having the properties (V) and such that $\lim_{x \to \infty} u(x) = m_0$. This solution is unique and it satisfies also the integral equation

$$u(x) = m_0 + (-1)^{n+1} \int_{x}^{\infty} \frac{(x - t)^{n-1}}{(n - 1)!} B(t, f(t))\, u(t)\, dt.$$

The existence of a number $k_0 > m_0$ such that $TG_{k_0} \subset G_{k_0}$ is assured by a convinient choice of x_0. The rest of the proof is nearly the same as in the proof of the theorem 5.

Also the proof of the following theorem is analogous [2].

Theorem 7. 1° *Let $P(x, \mathbf{u})$ be a function defined and continuous on Ω and non-negative in such a way that for every point $\mathbf{c} = (c_0, c_1, \ldots, c_{n-1}) \neq (0, 0, \ldots, 0)$ the function $P(x, \mathbf{c})$ is identically zero on none of the subintervals of the interval (a, ∞).*

2° *Let be $P(x, \mathbf{u}) \leqq B(x, \mathbf{u})$ for every point $(x, u) \in \Omega$ and let the function $B(x, \mathbf{u})$ fulfil all the conditions of the theorem 5 respectively 6.*

Then for the equation

$$y^{(n)} + (-1)^{n+1}P(x, y, y', \ldots, y^{(n-1)})\, y = 0 \tag{P}$$

hold all the statements of the theorem 5 respectively 6.

Let us return now to the theorems 1—4 which have established in the case of space S_{n-1}. But we can prove the validity of those theorems also in the case of other Banach spaces then S_{n-1} [2]:

Let $X \subset A_{n-1}$ be a Banach space with the norm $\| \ \|_X$ such that the convergence according to this norm implies also the q-convergence. Then the theorem 1 holds if we substitute S_{n-1} by X. If the q-compactness of the set $M \subset X$ follows from the properties that the functions of M are uniformly bounded and equicontinuous, the theorem 2 and 4 hold for X. If yet from the fact that the functions of the set M are uniformly bounded and equicontinuous follows that they are also bounded in the sense of the norm $\| \ \|_X$, the theorem 3 holds if we substitute S_{n-1} by X.

We are giving now some examples in which the above exposed ideas find their application.

Theorem 8. [7] *Let $B(x, \mathbf{u})$, $F(x, \mathbf{u})$, $\mathbf{u} = (u_0, u_1, \ldots, u_{n-1})$, $n \geqq 1$, be the functions non-decreasing in every of his variables u_i, $i = 0, 1, \ldots, n-1$ and such that*

$$|B(x, \mathbf{u})| \leqq F(x, \mathbf{u}) \quad \text{on} \quad \Omega. \tag{3}$$

Let K be a positive number, $x_0 > a$ and $0 \leqq k \leqq n-1$ an integer. Let

$$\varphi(x) = K \sum_{s=0}^{k} \frac{(x - x_0)^s}{s\,!}, \tag{4}$$

$$\int^{\infty} x^{n-k-1}F(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x), K, K, \ldots, K)\, \mathrm{d}x < \infty \tag{5}$$

for every $K < 0$,

$$\lim_{K \to \infty} \frac{1}{K} \int^{\infty} (x - x_0 + 1)^{n-k-1}F(x, \varphi(x), \ldots, \varphi^{(k)}(x), K, \ldots, K)\, \mathrm{d}x = 0 \tag{6}$$

Let be finaly $c_0, c_1, \ldots, c_k$ real arbitrary numbers. Then the differential equation

$$y^{(n)} + B(x, y, y', \ldots, y^{(n-1)}) = 0 \tag{E}$$

has at least one solution defined on $J = \langle x_0, \infty)$ and satisfaying the conditions

$$(7) \qquad y^{(i)}(x_0) = c_i, \qquad i = 0, 1, \ldots, k-1$$
$$\lim_{x\to\infty} y^{(k)}(x) = c_k,$$
$$\lim_{x\to\infty} y^{(i)}(x) = 0, \qquad i = k+1, \ldots, n-1.$$

I am going to scatch the proof. By a simple calculus we can see that the solution of the integral equation

$$(8) \quad y(x) = \sum_{s=0}^{k} c_s \frac{(x-x_0)^s}{s\,!} - \sum_{s=0}^{k-1} \frac{(x-x_0)^s}{s\,!} \int_{x_0}^{x} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} B(t, \mathbf{y}(t))\,\mathrm{d}t + $$
$$+ \sum_{s=k}^{n-1} \frac{(x-x_0)^s}{s\,!} \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} B(t, \mathbf{y}(t))\,\mathrm{d}t$$

is also the solution of the equation (E) and fulfils the conditions (7).

We are looking for this solution in the Banach space $C_{n-1,k} \subset A_{n-1}$ of all functions which have the bounded derivatives of the orders $k, k+1, \ldots, n-1$ on the interval $J = \langle x_0, \infty)$. Let the norm in $C_{n-1,k}$ be

$$||f(x)|| = \max_{k \leqq i \leqq n-1} \{\sup_J |f^{(i)}(x)|\} + \sum_{i=0}^{k-1} |f^{(i)}(x_0).$$

It can be easily shown that the convergence according this norm implies the q-convergence.

Let be $G_K = \{f(x) \in C_{n-1,k} \mid ||f(x)|| \leqq K\}$. Then for every $f(x) \in G_K$ holds

$$|f^{(i)}(x)| \leqq \varphi^{(i)}(x), \qquad i = 0, 1, \ldots, k,$$
$$|f^{(i)}(x)| \leqq K, \qquad i = k+1, \ldots, n-1.$$

If we respect (3), (5) and the monotony of $F(x, \mathbf{u})$ we have

$$(9) \qquad |B(x, \mathbf{f}(x))| \leqq F(x, \boldsymbol{\varphi}(x), \mathbf{K}),$$

where $F(x, \boldsymbol{\varphi}(x), \mathbf{K}) = F(x, \varphi(x), \varphi'(x), \ldots, \varphi^{(k)}(x), K, \ldots, K))$ and

$$(10) \quad \left| \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} B(t, \mathbf{f}(t))\,\mathrm{d}t \right| \leqq \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)\,!} F(t, \boldsymbol{\varphi}(t), \mathbf{K})\,\mathrm{d}t < \infty$$
$$s = k, k+1, \ldots, n-1.$$

This allows us to define the operator T on G_K by the formula

$$(11) \qquad Tf(x) = v(x) = \sum_{s=0}^{k} c_s \frac{(x-x_0)^s}{s\,!} -$$

$$-\sum_{s=0}^{k-1} \frac{(x-x_0)^s}{s!} \int_{x_0}^{x} \frac{(x_0-t)^{n-s-1}}{(n-s-1)!} B(t, \mathbf{f}(t))\, dt +$$
$$+\sum_{s=k}^{n-1} \frac{(x-x_0)^s}{s!} \int_{x}^{\infty} \frac{(x_0-t)^{n-s-1}}{(n-s-1)!} B(t, \mathbf{f}(t))\, dt.$$

We can see immediately that $TG_K \subset C_{n-1,k}$. From the condition (6) follows the existence a certain number K_0 such that $TG_{K_0} \subset G_{K_0}$. The q-continuity of T on G_{K_0} follows from the conditions (3), (5), from the monotony of $F(x, \mathbf{u})$ and from the theorem of Lebesgue.

Note $TG_{K_0} = H$. Let $H^{(i)}$, $i = 0,1, \ldots, n-1$ be the set of the derivatives of the order i of all the functions of H. It can be proved that the functions of $H^{(i)}$, $i = k, k+1, \ldots, n-1$ are uniformly bounded and equicontinuous on J. Le us make the closure of the convexe hull $\overline{\widehat{H}} = M$ of H. We can prove that the functions of $M^{(i)}$, $i = k, k+1, \ldots, n-1$ are uniformly bounded and equicontinuous on J. From this we can prove that M is q-compact. Then we now that M is convexe, closed, q-compact and $M \subset G_{K_0}$. From this last relation we obtain that $TM \subset TG_{K_0} = H \subset \overline{\widehat{H}} = M$. The application of the theorem 1 finishes the proof.

If we take for $F(x, \mathbf{u})$ a linear expression,

$$F(x, \mathbf{u}) = a(x) + \sum_{i=0}^{n-1} a_{n-i}(x)\, |u_i|,$$

we obtain from the theorem 8 the

Theorem 9. *Let be* $|B(x, \mathbf{u})| \leqq a(x) + \sum_{i=0}^{n-1} a_{n-i}(x)\, |u_i| = F(x, \mathbf{u})$ *for every* $(x, \mathbf{u}) \in \Omega$ *and let be* $a(x) \geqq 0$, $a_{n-i}(x) \geqq 0$,

$\int^{\infty} x^{n-k-1} a(x)\, dx < \infty$, $\int^{\infty} x^{n-k-1} a_{n-i}(x)\, dx < \infty$, $i = k+1, k+2, \ldots, n-1$,

$\int^{\infty} x^{n-i-1} a_{n-i}(x)\, dx < \infty$, $\quad i = 0,1, \ldots, k$.

Then for x_0 sufficiently large the affirmations of the theorem 8 hold.

The theorem 8 gives the results which are a generalization of the results of M. P. Waltman [4] for the equation $y^{(n)} + f(x, y) = 0$. He proves, by a different way, the existence of a solution $y(x)$ for which $\lim_{x \to \infty} y(x)/x^{n-1} = \beta \neq 0$ under the conditions: $|f(x, y)| \leqq a(x)\, y^{\alpha}$, $\alpha > 0$ and $\int^{\infty} x^{\alpha(n-1)} a(x)\, dx < \infty$.

By the same method as above we can prove the following theorems (in the space $C_{n-1,n-1}$): [3].

Theorem 10. *Let $B(x, \mathbf{u})$, $F(x, \mathbf{u})$ be the continuous functions in the domain Ω. Let $F(x, \mathbf{u})$ be non-decreasing in every of the variables u_i, $i = 0,1, \ldots, n-1$. Let $K > 0$, $x_0 > a$ and*

$$\varphi(x) = K \sum_{s=0}^{n-1} \frac{1}{s!} (x - x_0)^s ,$$

(12) $$\int^{\infty} x^{n-1} F(x, \boldsymbol{\varphi}(x))\, \mathrm{d}x < \infty \qquad \text{for every } K > 0,$$

(13) $$\lim_{K\to\infty} \frac{1}{K} \int^{\infty} x^{n-1} F(x, \boldsymbol{\varphi}(x))\, \mathrm{d}x = 0,$$

(14) $$|B(x, \mathbf{u})| \leqq F(x, \mathbf{u}) \qquad \text{for every } (x, u) \in \Omega.$$

Finaly, let $c_0, c_1, \ldots, c_{n-1}$ be arbitrary real numbers.

Then the equation (E) has at least one solution $u(x)$, which exists on $J = \langle x_0, \infty)$ and for which hold the formulae:

(15) $$u^{(i)}(x) = \sum_{s=i}^{n-1} c_s \frac{(x - x_0)^{s-i}}{(s-1)!} + o(1), \qquad i = 0,1, \ldots, n-1.$$

Note. If we substitute the condition (12) by

$$\int^{\infty} x^{n-1+\varepsilon} F(x, \boldsymbol{\varphi}(x))\, \mathrm{d}x < \infty, \qquad \varepsilon > 0,$$

then in the formulae (15) it is possible to substitute $o(1)$ by $o(x^{-\varepsilon})$. And if we take for $F(x, u) = a(x) + \sum_{i=0}^{n-1} a_{n-i}(x)\, |u_i|$, theorem 10 gives a generalization of the results of M. ZLÁMAL [5] found for the linear differential equations.

Theorem 11. [6] *Let fulfil all the conditions of the theorem 10 with the exception of the conditions (12) and (13), which will be substitued by*

(12′) $$\int^{\infty} F(x, \boldsymbol{\varphi}(x))\mathrm{d}x < \infty \qquad \text{for every } K > 0,$$

(13′) $$\lim_{K\to\infty} \frac{1}{K} \int^{\infty} F(x, \boldsymbol{\varphi}(x))\, \mathrm{d}x = 0.$$

Then by the initial conditions $(x_0; c_0, c_1, \ldots, c_{n-1})$, where c_i are real arbitrary numbers, is determined at least one solution $u(x)$ of (E) which exists on $J = \langle x_0, \infty)$.

If morover the condition (12) is satisfied, then for this solution $u(x)$ hold the formulae:

$$u^{(i)}(x) = \sum_{s=i}^{n-1} c_s \frac{(x - x_0)^{s-i}}{(s - i)!} + O(1), \qquad i = 0, 1, \ldots, n - 1. \tag{16}$$

Those are some exemples where *I* profit with success of the variants of the theorem of SCHAUDER mentioned above.

I wish yet to remark that one can utilise this method in many other cases, chiefly in the cases where the theorem of Arzela has been applied and that's why it has been limited on a finite interval. In the first place that are the problems of the global existence, the linear problems and the boundary-value problems.

The notions and the theorems of which *I* spoke, have been prepared in to their application to the problems of the differential equations of n-th order. There is no difficulty to adapt those for the systems of differential equations.

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